Prop: If $H \leq G$ is a closed subgp then there exists a rep $\rho: G \to GL(V)$ and a line $L \subseteq V$ s.t. $H = N_G(L)$

Pf: Consider the action $G \curvearrowright k[G]$ let $I \triangleleft k[G]$ s.t. $k[H] = k[G]/I$. Follows from prior arguments that $h \in G(R)$ is in $H(R)$ if and only if $hI \subseteq I$.

may fill this in later

$k[G]$ is f.g. $\Rightarrow$ so Noeth $\Rightarrow I$ is f.gen., say $I = \langle x_1, \ldots, x_n \rangle$

So we can choose a f.d'l subrep of G $(W \subseteq V)$ contg $x_1, \ldots, x_n$. So it follows that $h \in G(R)$ is in $H(R)$ if and only if $h\langle x_1, \ldots, x_n\rangle \subseteq I$

$$h\langle x_1, \ldots, x_n\rangle \subseteq I \cap W_R$$

$$h(I \cap W_R) \subseteq I \cap W_R$$

So $H = N_G(I \cap W)$ in action on W

if $m = \dim I \cap W$ then G acts on $\Lambda^m W$ and (as a variety action) on

$$\mathbb{P}(\Lambda^m W) \ni \Lambda^m U = k \cdot u_1 \wedge \cdots \wedge u_m$$

$$\cup \qquad\qquad \uparrow \text{Plüker embedding}$$

$$Gr(m, W) \ni U = \langle u_1, \ldots, u_m \rangle$$

H = stuff that preserves $I \cap W$ in W

= " " " " point corresp. to $I \cap W$ in $Gr(m, W)$

= " " " " line $\Lambda^m(I \cap W)$ in $\mathbb{P}(\Lambda^m W)$ $\square$

Some translations:

$H = N_G(W \cap I)$

language for talking about general action of G on $\mathbb{A}(W)$

$$H(R) \star \{ g \in G(R) \mid \forall S/R,\ g_S \cdot (W\cap I)_S \subseteq (W \cap I)_S \}$$

$$\star \{ g \in G(R) \mid g\cdot (W \cap I) \underset{=}{\subseteq} (W\cap I) \}$$

because action is lin.

$$Gr(m,W)(R) = \{ [P] \mid P \subseteq W,\ P \text{ proj. rank } m \text{ over } R \text{ s.t. } \exists U \text{ s.t. } P \oplus U = W \}$$

Observe that $G \circlearrowright W$ (rep) $\Rightarrow$ G acts on variety $Gr(m,W)$

$G \times Gr(m,W) \longrightarrow Gr(m,W)$ nat. tran. of functors

$G(R) \times Gr(m,W)(R) \to Gr(m,W)(R)$

$(g,[P]) \longmapsto [gP] = [\{ gp \mid p \in P\}] \in Gr(m,W)(R)$

U comp of P $\rightsquigarrow$ gU comp. of gP

$\Rightarrow G \circlearrowright Gr(m,W)$

by def of $H(R)$ above, can see that

$$H(R) \star \{ g \in G(R) \mid g\cdot[P] = [P] \}$$
$$(gP = P)$$

Def A smooth LAG$/_k$ G is unipotent if all $g \in G(\bar{k})$ are unipotent

(i.e. $g_{ss} = 1$ all $g \in G(\bar{k})$)

$g_{ss} = g_{sep}$

Jordan decomp $\Rightarrow$ given a faithful rep $G \hookrightarrow GL(V)$

g unip $\Rightarrow$ Jordan form has 1's along diagonal

$g = 1 + T$ where T is upper Δ.

Prop G unip, $G \curvearrowright GL(V)$ any rep, $\exists$ basis for V making all $g \in G(\bar{k})$ simult. of form $1+T$, T upper Δ.

Follows from:

Lemma: G unipotent, $G \curvearrowright GL(V)$, then $\exists$ $v \in V$ fixed by all $g \in G(\bar{k})$.

If lemma true, induct on $\dim V$ to prove Prop:

$G(\bar{k}) \cdot v$

Warning: consider the gp $R_{L/k}\mathbb{G}_{m,L}$ where $L = k(\sqrt[p]{a})$
$a \notin k^p$ is char $k = p$.

Def: if A is any (not nesc. comm.)
f.d'l k-alg, $\mathbb{G}_{m,A}(R) = (A \otimes_k R)^\times$ multiplicatively.

ex: $GL(V) = \mathbb{G}_{m,\mathrm{End}(V)}$

Def: If X is any L-functor, L/k comm. k-alg.
$$R_{L/k}(X)(R) = X(R \otimes_k L)$$ "Weil restriction of scalars"

Ex: $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) = \{(a,b) \mid a+ib \in \mathbb{C}^\times\} = \{(a,b) \mid a^2+b^2 \neq 0\}$
$R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})(\mathbb{R}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) = \mathbb{C}^\times$

$\mathbb{G}_{m,A}$ $\qquad$ $\mathbb{G}_{m,L}$ $\qquad$ $\begin{matrix} L \\ | \\ k \end{matrix}$
k-gp
or L-gp!

$R_{L/k}\mathbb{G}_{m,L/L} = \mathbb{G}_{m,L/k}$

$\begin{matrix} L \\ |\ p\text{-insep} \\ k \end{matrix}$ $\qquad$ $\mathbb{G}_{m,L} = R_{L/k}\mathbb{G}_m$
as a k-gp acts as $L^\times$

get reps of $\mathbb{G}_{m,k}$ from k-vectspaces.

irred reps. which aren't diagonalizable are

$T \circlearrowright k[x]/\min_T = k[x]/x^p - a$

mult. by $\sqrt[p]{a}$

over $\bar{k}$, if $\alpha^p = a$

$y = x - \alpha$

$$\frac{k[x]}{x^p - \alpha^p} = \frac{k[y]}{y^p}$$

$$\| $$

$$\frac{k[x]}{(x-\alpha)^p}$$