Last time

$L = k(\alpha)$ $\quad \alpha^p = a \in k \setminus k^p$ $\quad$ char $k = p$
$|$
k

$R_{L/k}\mathbb{G}_m = G \qquad G(k) = R_{L/k}\mathbb{G}_m(k) = \mathbb{G}_m(L) = L^*$

$= \{(a_0, \dots, a_{p-1}) \in k \mid \vec{a} \neq 0\}$

$a_0 + a_1\alpha + \dots + a_{p-1}\alpha^{p-1} \in L$

$$\boxed{\begin{array}{l} R_{L/k}X(R) = X(R \otimes_k L) \\ X/L \end{array}}$$

$(a_0, \dots, a_{p-1}) \cdot (b_0, \dots, b_{p-1})$

$= (a_0 b_0 + a(a_1 b_{p-1} + a_2 b_{p-2} + \dots + a_{p-1} b_1),$

$a_0 b_1 + a_1 b_0 + a(a_2 b_{p-1} + \dots),$

$\vdots$

$)$

$G(k)$ every element is semisimple

for $\beta \in G(k) \hookrightarrow$ "$\mathrm{End}_k(L)$" $= GL_p$

$\min_\beta(T) = \min$ of β as an elmt of field ext.

$L \overset{p}{|} k$ field of deg p, $\beta \in k$ $\quad \min_\beta = T - \beta$

$\beta \notin k \Rightarrow \min_\beta$ irred deg p

but $G(\bar{k}) =$ mult. gp of $L \otimes_k \bar{k}$

$\bar{k}[x]/x^p - a \qquad \alpha \in \bar{k}$

$= \bar{k}[x]/(x-\alpha)^p \qquad y = x - \alpha$

$\{a_0 + a_1 y + \dots + a_{p-1}y^{p-1} \mid a_0 \neq 0\} = \bar{k}[y]/y^p$

local rg w/ max'l ideal gen by Y

$$Y \mapsto \begin{bmatrix} 0 & 0 & & \\ 1 & 0 & & \\ 0 & 1 & \ddots & \\ \vdots & & \ddots & 0 \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} a_0 & & & & \\ a_1 & \ddots & & 0 & \\ a_2 & \ddots & \ddots & & \\ \vdots & & & & \\ & & a_2 & a_1 & a_0 \end{bmatrix}$$

$$a_0 = 1 \rightsquigarrow \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}$$ are all unipotent

Remark: the unipotent elements in a gp are naturally a closed subscheme.

$G \hookrightarrow GL_n$ **Define** $\mathcal{U}_G$ subscheme defined by eqn: $(X-1)^n = 0$

$X \in G \hookrightarrow GL_n \hookrightarrow M_n = \mathbb{A}^{n^2}$

use radical of this ideal to make $\mathcal{U}_G$ reduced.

unip. elmts of $G(L) = \mathcal{U}_G(L)$

Today:

- Structure of comm. LAG's over perfect fields
- Structure of unipotent gps
- Solvable gps

Structure of comm LAG's / k perfect

G comm. LAG.

<u>recall</u>: if $h, g \in G(k)$ $\overset{\text{commute \&}}{\vee}$ are both unip. or both sep.

then so is hg, g^{-1}

Since k perfect, separable = semisimple & any $g \in G(L)$

L/k alg. field ext, $g = g_s g_u$ uniquely w/ g_s sep, g_u unip.

In this case $\underset{\overset{\shortparallel}{G_u}}{\mathcal{U}_G}$ as above is a closed subgp of G

<u>Define</u>: G_s = Zariski closure of $\{g \in G(\bar{k}) \mid g \text{ is sep}\}$

<u>what I mean</u> G_s is cut out by an ideal in $k[G]$ s.t. all pts desired over $\bar{k}$ here are in zero locus of this ideal

<u>Can think of this as</u>: $\operatorname{Spec} \bar{k} \xrightarrow{g} G$

g sep.

looking at Zar. closure of union of all images as above.

Claim: $G_s(\bar{k})$ consists of sep elements only.

<u>Pf</u>: WLOG can consider $k = \bar{k}$ then consider all sep elements $S \subset \underset{\overset{\wedge}{GL_n}}{G(\bar{k})}$

these consist of an (infinite) collection of commuting diagonalizable matrices in GL_n $\Longrightarrow$ simult. diag.

Choose a basis $\mathcal{B}$ so that $S \subseteq \underset{\text{diag.}}{\overset{\shortparallel}{D_n}} \subseteq GL_n$

$\overline{S} \subseteq D_n$ = closed in GL_n $\Rightarrow$

$\overline{S}$ is still sep. $\square$.

Observe that decom. gives surjectivity of $G_s \times G_u \xrightarrow{\sim} G$ (over $\bar{k}$)

$g_s, g_u \longrightarrow g_s g_u$

uniqueness $\Rightarrow$ inj

Thm

$G \simeq G_s \times G_u$ G comm. / perfect

where G_s & G_u are closed subgps whose elements are sep & unip. respectively

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Structure of unipotent gps

Thm: If G is unipotent over a perfect field k $G \hookrightarrow GL_n$ then we can find a basis where elements of G are simult. upper Δ.

Lemma: If $\rho: G \longrightarrow GL(V)$ is any rep. of unipotent / perfect k then $\exists$ $v \in V \setminus \{0\}$ which is G-fixed (i.e. $gv = v$ for all $g \in G(\bar{k})$)

<u>Lemma</u>: if $\mathcal{G} \subseteq GL(V)$ is any <u>abstract</u> subgp where V/k v.s. $k = \bar{k}$

s.t. V is irred (simple) as a rep of $\mathcal{G}$ then

$End(V)$ is gen. as a k-alg by $\mathcal{G}$.

(Recall: V irred $\iff$ if $W \subseteq V$ is $\mathcal{G}$-stable ($\mathcal{G}W \subseteq W$) then $W = 0$ or V.)

<u>Thm "Jacobson Density"</u>

If R ring M a left R-mod, then M is an $\underset{\substack{\| \\ E}}{End_R(M)}$-module

and also a $D = End_E(M)$ module.

map $R \xrightarrow{\varphi} D$

$r \longmapsto [m \mapsto rm]$

then φ has <u>dense image</u>

(if $m_1, \ldots, m_s \in M$ and $d \in D$ then $\exists\, r \in R$ s.t. $rm_i = dm_i$ all i)

in particular if all things are v.spaces/algs/k and M f.d'l/k then $R \twoheadrightarrow D$

<u>Thm (Schur's Lemma)</u>

Let R a ring, $M \neq 0$ a simple R-module. $E = End_R(M)$

then E is a division ring.

Pf: if $f \in E$, $\operatorname{im} f \subseteq M$ $\ker f \subseteq M$
submods if $f \neq 0$ then $\operatorname{im} f \neq 0$
$-$ $\operatorname{im} f = M$
$\Rightarrow \ker f \neq M \Rightarrow \ker f = 0$
$\Rightarrow f$ is an iso.
inverse also an End $\Rightarrow$ D.]

Pf: let $R = k$-alg. gen by $\mathcal{G} \subseteq GL(V) \subseteq End(V)$
V simple as $\mathcal{G}$ rep ($\mathcal{G}W \subseteq W \Rightarrow W = 0$ or V)
$\Rightarrow$ ($RW \subseteq W \Rightarrow W = 0$ or V)
$\Rightarrow V$ is simple R-module. $\Rightarrow E = End_R(V) \subseteq End_k(V)$ is div. ring.
also a f.dim'l k-alg.
where $k = \bar{k}$
$\Rightarrow E = k$
(if $d \in E \setminus k$ $k[d]$ f.d. domain$/k \Rightarrow$ field)
$k = \bar{k} \Rightarrow k[d] = k$

$D = End_E(V) = End_k(V)$ & $R \twoheadrightarrow D = End_k(V)$ ◻.
J. density

Lem: if $T \in M_n(k)$ then T is determined by map
$S \longmapsto tr(TS) \in k$
$M_n(k) \longrightarrow k$

Strategy for find $v \in V$ s.t. $Gv = v$

Can assume V is irred,

else, choose $\underset{\neq 0}{W} \subseteq V$ find fixed in W.

show V irred $\Rightarrow V \simeq k$ w/ triv. rep.

$\rho : G \longrightarrow GL(V)$ $\quad \rho(g) = 1$ all g.

$$g = 1 + g_n \qquad g_n = 0$$

$tr(g_n \cdot h) = 0$ all $h \in G(\overline{k})$

$\Rightarrow tr(g_n \cdot End(V)) = 0$

$\Rightarrow g_n = 0.$