Structure of unipotent gps

Thm: If G is unipotent over a perfect field k, $G \hookrightarrow GL_n$
then we can find a basis where elements of G are simult. upper Δ.

Lemma: If $\rho: G \to GL(V)$ is any rep. of unipotent / perfect k
then $\exists\, v \in V \setminus \{0\}$ which is G-fixed (i.e. $gv = v$ for all $g \in G(\bar{k})$)

Lemma $\Rightarrow$ Thm ?

By induction on $\dim V$ $\qquad \dim 1 \Rightarrow$ rep = trivial

Induction step if v fixed by G then $V/\langle v \rangle = W$

then W also a rep (quotients of reps are reps)

By induction, rep on W is lower Δ w/ 1s on diag.

$$\begin{array}{c} v \\ W \end{array} \begin{pmatrix} \boxed{1} & * \\ 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{matrix} \end{pmatrix} \qquad \checkmark$$

Pf of Lemma: To find a fixed vector in V can assume V irred.

In fact: in this case we'll show V is a trivial rep.

Choose $g \in G(\bar{k})$ we'll show $\rho(g) = id_V$

By "linear algebra" can assume $k = \bar{k}$ — point is to show that the eigenspace w/ eigenvalue 1 is $\neq 0$.

Want to show $\rho(g) = 1$ will do this using $g = g_s + g_n$ in $\mathrm{End}(V)$

$= 1 + g_n$

Want: $g_n = 0$

$g_n = 0 \iff tr(\rho(g_n) T) = 0$ all $T \in \mathrm{End}(V)$

But: if R = subalg gen by $\rho(G(k))$ in $\mathrm{End}(V)$

then Jacob. density $\Rightarrow R = \mathrm{End}(V)$ (irreducibility)

$kG(k) \xrightarrow{\rho} \mathrm{End}(V)$ so can consider T of form $\Sigma x_g \rho(g)$

$\| \; \left\{ \sum_{g \in G(k)} x_g g \;\middle|\; x_g \in k \right\}$

enough to show $tr(\rho(g_n)\rho(h)) = 0$ all $h \in G(k)$

$\|$

$tr((\rho(g) - 1)\rho(h)) = tr(\rho(g)\rho(h) - \rho(h))$

$= tr(\rho(g)\rho(h)) - tr(\rho(h))$

$= tr(\rho(gh)) - tr(\rho(h))$

$tr \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} - tr \begin{bmatrix} 1 & & \\ & \ddots & \\ * & & 1 \end{bmatrix}$

$= \dim V - \dim V = 0.$

$\Rightarrow \rho(g_n) = 0$ all g. $\Rightarrow \rho(g) = 1 + \rho(g_n)$

$= 1$ $\square$.

So far:

Commutative structure thm: G/k perfect comm $\Rightarrow$ $G \cong G_s \times G_u$

comm. all elmts semisimple (sep) — G_s; unipotent — G_u.

Unipotent/perfect = simult. unipotent matrices.

Remark: if $G/k = \bar{k}$ comm, $G_s \overset{\rho}{\hookrightarrow} GL(V)$ is diagonal after some change of basis of V.

Since each $g \in G_s$ $\rho(g)$ separable as a lin trans $\Rightarrow$ diagonalizable but also G comm $\Rightarrow$ simultaneously diagonalizable $\square$.

Fact (will come back to) in this case G_s^0 connected comp. of 1 we have $G_s^0 \cong \mathbb{G}_m^n$ some n $(k = \bar{k})$

Next target: Solvable groups

$$\begin{pmatrix} x & & 0 \\ & \ddots & \\ * & & x \end{pmatrix}$$

Method: define derived subgp (commutator subgp)

"$DG = [G,G]$" $G \supseteq D(G) \supseteq D(D(G)) \supseteq \cdots \supseteq \ldots \overset{?}{e}$ solvable.

(Conrad 16.2)

Prop: Let G be a smooth k-gp and $X_1, \ldots, X_n \neq \emptyset$ a collection of varieties (geom. integral finite type k-schemes) and $f_i : X_i \longrightarrow G$ morphisms. $e \in f_i(X_i)$

... connected

then $\exists!$ smooth closed connected k-subgp $H \leq G$ s.t.

$$H(\bar{k}) = \langle X_1(\bar{k}), \dots, X_n(\bar{k}) \rangle$$

and $\exists$ finite list of pairs $(i_1, m_1), (i_2, m_2), \dots, (i_r, m_r)$

s.t. $X_{i_1} \times \cdots \times X_{i_r} \to H$ via

$$(x_1, \dots, x_r) \longmapsto x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \quad \text{is surjective.}$$

Proof strategy

keep adding X_i's to make dimension of products get bigger

$$(\mathrm{im}\, X_1(\bar{k}))(\mathrm{im}\, X_2(\bar{k}))(\mathrm{im}\, X_5(\bar{k}))^{-1} \cdots$$

keep going to grow dimension of image.

U product so far, can replace U by ~~UU^{-1} to assume $e \in U$~~

once growth stops, let H = closure of $\underbrace{\text{product so far}}_{U}$.

~~$U \subseteq UU$~~

$U \subseteq X_i U$

UX_i

$U \subseteq UU^{-1}$ dense so $UU \subseteq \bar{U} = H$

$UU^{-1} \subseteq \bar{U} = H$

$$\begin{tikzcd} U \times U \arrow[r] \arrow[d, phantom, "\cap"] & H \arrow[d, phantom, "\cap"] \\ H \times H \arrow[r] \arrow[ur, dashed] & G \end{tikzcd}$$

but in fact didn't need closure!

$UU^{-1} = H$

lemma: if $U \subseteq H$ (gp variety) dense in a LAG then

$H = UU^{-1}$ (on level of $\bar{k}$)

Pf: WTS: $\forall h \in H$, $h \in UU^{-1} \Leftrightarrow hU \cap U \neq \emptyset$

$h = uv^{-1}$ $\quad$ $hv = u$

but hU & U are both dense open

$\Rightarrow hU \cap U \neq \emptyset$ $\square$.

$$DG = \langle \operatorname{im}: G \times G \to G \rangle$$
$$g, h \mapsto ghg^{-1}h^{-1}$$