$\exists$ course website

dkrashen.org/.... teaching, this class

register email, after will poll for office hours

today OH 4-5pm

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Last time:

Def k-space: functor from (commutative) k-algebras to Sets

Def Affine k-scheme: representable functor (space)

$$X : R \longmapsto \mathrm{Hom}_{k\text{-alg}}(A, R) \quad \text{some fixed } A$$

A is the coordinate ring of X $\quad A = k[X]$

X is the spectrum of A $\quad X = \mathrm{Spec}\, A$

Def A f.gen (as an algebra), $X = \mathrm{Spec}\, A$ is called a finite type affine k-scheme.

$$k[x_1, \ldots, x_n] \twoheadrightarrow A$$

$$\frac{k[x_1, \ldots, x_n]}{I} \simeq A \quad \text{Noeth} \Rightarrow I = (f_1, \ldots, f_m)$$

$$X(R) = \mathrm{Hom}_k(A, R) = \{\varphi \in \mathrm{Hom}_k(k[x_1, \ldots, x_n], R) \mid \varphi(f_i) = 0\}$$

$$= \{(a_1,\dots,a_n) \in R^n \mid f_i(a_1,\dots,a_n) = 0, \text{ each } i\}$$

$$\varphi : x_i \mapsto a_i \qquad \varphi(f_i) = f_i(a_1,\dots,a_n)$$

$$A = k[x,y]/x \qquad k[y]$$

$$R \mapsto \{(a,b) \in R \mid a = 0\} \qquad R \mapsto \{a \in R\} = R$$

$$Z(f_1,\dots,f_m) \subseteq \mathbb{A}^n \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Z(g_1,\dots,g_e) \subseteq \mathbb{A}^r$$

$$\varphi_1,\dots,\varphi_r \in k[x_1,\dots,x_n]$$

$$\psi_1,\dots,\psi_n \in k[x_1,\dots,x_r]$$

"exercise"
nat transformations of spaces $\longleftrightarrow$ morphisms of affine schemes

<u>Def</u> if $X$ is a space, $|X|$ for its equivalence classes of field points.

(Recall if $X = \operatorname{Spec} A$ $\quad |\operatorname{Spec} A| =$ prime ideals of $A$)

<u>Recall</u>: $\sqrt{R} = \{a \in R \mid a^n = 0 \text{ some } n\}$

$$= \bigcap_{\mathfrak{p} \in |\operatorname{Spec} R|} \mathfrak{p}$$

$|\operatorname{Spec} A| = |\operatorname{Spec} A/\sqrt{A}|$

Def: $A$ is reduced if $\sqrt{A}=0$, $A/\sqrt{A}$ is always reduced.

Def $X=\operatorname{Spec} A$ affine scheme $I \triangleleft A$ ideal

Defn: $Z(I) \leq X$ (i.e. for all $R$, $Z(I)(R) \subseteq X(R)$)

$$Z(I)(R) = \{ \underset{\substack{\shortparallel \\ \text{"}(r_1,\dots,r_n)\text{"}}}{x} \in \operatorname{Hom}(A,R) \mid x(f)=0, f \in I \}$$

$$\shortparallel$$

$$\operatorname{Hom}(A/I, R)$$

"closed subfunctors"

observe: $Z(I) = \operatorname{Spec} A/I$ $\quad Z(I) \subseteq \operatorname{Spec} A$

and $|Z(I)| = |\operatorname{Spec} A/I| \leq |\operatorname{Spec} A|$

$$\shortparallel$$

$$\{ \mathfrak{p} \in |\operatorname{Spec} A| \mid I \subseteq \mathfrak{p} \}$$

Comment: lots of different $I$'s w/ same $|Z(I)|$

$k[x]/x \quad k[x]/x^2$ same $|Z(I)|$

$I=(x) \quad I=(x^2)$ different $Z(I)$

What about $U(I) = X \setminus Z(I)$ ? $\quad X = \operatorname{Spec} A$

Naive guess: $\mathcal{U}(I)(R) = \{\varphi : A \to R \mid \varphi(f) \neq 0 \text{ some } f \in I\}$

Problem: this isn't a subfunctor

$$
\begin{tikzcd}
\mathcal{U}(I)(R) \arrow[r, hook] \arrow[d, "\times" marking] & X(R) = \operatorname{Hom}(A,R) \arrow[d] \\
\mathcal{U}(I)(S) \arrow[r, hook] & X(S) = \operatorname{Hom}(A,S)
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] & \neq 0 \arrow[dd, bend left, mapsto] \\
A \arrow[r] \arrow[dr, dashed] & R \arrow[d] \\
& S \quad 0
\end{tikzcd}
$$

Def: presheaves $F \subseteq G$ means $F(R) \subseteq G(R)$ all $R$.

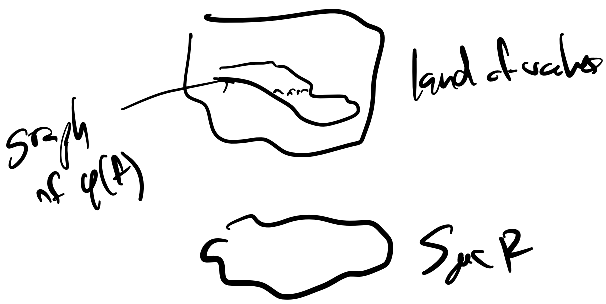
dfn: largest sub-presheaf of $F$ which is a subfunctor

= inner subfunctorification.

$$x \in X(R) = \operatorname{Spec} A(R) = \operatorname{Hom}_k(A,R)$$

$$= \operatorname{Hom}_{k\text{-spaces}}(\operatorname{Spec} k, \operatorname{Spec} A)$$

$f$




$$
\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "f"] & \{\text{values}\} \\
\operatorname{Spec} R \arrow[u, "x = \varphi"] \arrow[ur, dashed, "\varphi(f)"'] &
\end{tikzcd}
$$

$r \in R$ "nearly zero"

$$\forall\ R \xrightarrow{\varphi} S \neq 0 \qquad \varphi(r) \neq 0$$

Claim: $r$ is nearly zero $\iff r \in R^*$.

$R \to R/rR$ ✓

Q: what's the ideal version?

A: $I = R$.

Ex: $\mathcal{U}(f) = \operatorname{Spec} A[f^{-1}]$

Observations: $|Z(I+J)| = |Z(I)| \cap |Z(J)|$

$|U(I+J)| = |U(I)| \cup |U(J)|$

$I = (f_1, \ldots, f_n) = (f_1) + (f_2) \cdots + (f_n)$

$|U(I)| = \bigcup |U(f_i)|$

$|U(I)|$ form a topology, $|U(f)|$ basis.

"Zariski top on Spec A"

**Def** A group space is a functor

$$k\text{-alg} \longrightarrow \text{groups}.$$

**Def** if $\mathcal{C}$ is a category w/ products and a final object $*$ then a group object in $\mathcal{C}$ is:

an object $G \in \text{ob}\,\mathcal{C}$ together w/ morphisms

$$m : G \times G \longrightarrow G$$

$$e : * \longrightarrow G$$

$$\iota : G \longrightarrow G$$

s.t.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times id_G"] \arrow[d, "id_G \times m"'] \arrow[dr, phantom, "\circlearrowright"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

"associativity"

$$G \times * \simeq G \simeq * \times G$$

$$g \longmapsto (g, *)$$

$$\begin{tikzcd}
G \arrow[r, "\sim"] \arrow[d, "\wr"'] \arrow[ddr, "id_G"] & G \times * \arrow[d, "id_G \times e"] \\
* \times G \arrow[d, "e \times id_G"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

$(g, e)$

identity

$$\begin{tikzcd}
G \arrow[r, "id_G \times \iota"] \arrow[d, "\iota \times id_G"'] \arrow[dr, bend left] & G \times G \arrow[dd, "m"] \\
G \times G \arrow[dr, "m"'] & * \arrow[d, "e"] \\
& G
\end{tikzcd}$$

inverses

Next time: A group space = gp object in spaces

a gp space which is affine scheme "group scheme"

= gp object in $(k\text{-alg})^{op}$

= Hopf algebra