Back to unipotent grps for a bit

Def G a smooth LAG over a field k is unipotent if all $g \in G(\bar{k})$ are unipotent

(i.e. if $G \hookrightarrow GL(V)$ faithful rep, then $g \in G(\bar{k})$ is represented by a unip. matrix (i.e. a matrix of form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$))

Showed if G unipotent over $k = \bar{k}$, G acts on V then V has a fixed vector - $\exists\, v \in V$ s.t. $gv = v$ all $g \in G(k)$

$\Rightarrow$ (via induction) can find a basis for V s.t. all $g \in G(k)$ are simult. $\underbrace{\text{upper } \Delta \text{ w/ 1's on diagonal}}_{\text{unipotent matrices}}$)

Conversely, if G is a smooth LAG $/ k = \bar{k}$ then G unipotent iff for each rep $G \circlearrowright V$, $\exists$ fixed vect $v \in V$.

Lemma: If G a smooth LAG $/ k = \bar{k}$ then TFAE:

1) G unipotent
2) $\exists$ faithful unipotent rep of G
3) every representation of G admits a fixed vector

Corollary

If $N \trianglelefteq G$ (closed normal) and $N, G/N$ unipotent, then so is G. $/ k = \bar{k}$ all smooth LAG

Pf: we check that if V is a rep, of G, $\exists$ fixed vectr.

Let $W = \{w \in V \mid nw = w \text{ all } n \in N(k)\}$

N unip. $\Rightarrow W \neq 0$.

W is a subrep since if $g \in G(k)$ $w \in W$ want to show $gw \in W$

but $n \cdot (gw) = gg^{-1}ngw = g(\underbrace{(g^{-1}ng)}_{\in N}w) = gw$

and now have a rep of G on W

but N acts trivially $\Rightarrow$ rep of G/N on $W \Rightarrow$ fixed vectr. . . . $\square$

Cor if G unip $N \trianglelefteq G$ as above so is N & G/N.

Solvable groups

(Conrad 16.2)

Prop: Let G be a smooth k-gp and $X_1, \dots, X_n$ $\neq \emptyset$ a collection of varieties (geom. integral finite type k-schemes) and $f_i : X_i \to G$ morphisms. $e \in f_i(X_i)$

then $\exists!$ smooth closed connected k-subgp $H \leq G$ s.t.

$H(\bar{k}) = \langle X_1(\bar{k}), \dots, X_n(\bar{k}) \rangle$

and $\exists$ finite list of pairs $(i_1, m_1), (i_2, m_2) \dots, (i_r, m_r)$

s.t. $X_{i_1} \times \dots \times X_{i_r} \to H$ via

$(x_1, \dots, x_r) \longmapsto x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}$ is surjective.

Def $\mathcal{D}G$ = closed smooth connected subgp gen by $[G,G]$

(G smooth connected CAG$/_k$)

$$G\times G \longrightarrow G$$
$$g,h \longmapsto ghg^{-1}h^{-1}$$

Fact here: at the level of points over $\bar{k}$, $\mathcal{D}G(\bar{k}) = [G(\bar{k}), G(\bar{k})]$
(from prop)

Def $G \supset \mathcal{D}G \supset \mathcal{D}^2G \supset \dots \supset$ $\quad \mathcal{D}^2 G = \mathcal{D}(\mathcal{D}G)$

note: this eventually stabilizes for dimensions

descending / lower central series

Def G solvable if eventually is $(e) = \mathcal{D}^nG$

Prop. G is solvable if $\exists$ a sequence of normal closed connected smooth subgps

$G = N_0 \supseteq N_1 \dots \supseteq N_n = (e)$ $\quad N_i \triangleleft G$ w/ N_i/N_{i+1} commutative.

Remark: $\mathcal{D}G(\bar{k}) = \underbrace{\mathcal{D}(G(\bar{k}))}_{\text{standard gp. theoretic one}}$

and $(\mathcal{D}^nG)(\bar{k}) = \mathcal{D}^n(G(\bar{k}))$ so G solvable $\Leftrightarrow$ $G(\bar{k})$ solvable as an abstract gp.

<u>Next goal</u> Borel fixed pt / Lie-Kolchin thms

<u>Will use fact:</u> if T is a commutative connected gp scheme consisting of only sep. elements (at $\bar{k}$) then T is a torus

i.e. $T_{\bar{k}} \cong \mathbb{G}_{m,\bar{k}} \times \cdots \times \mathbb{G}_{m,\bar{k}}$

<u>Def</u> G (smooth, connected, LAG) is split-solvable if $\exists$ sequence of normal closed connected smooth subgps

$$G = N_0 \geq \cdots \geq N_n = (e)$$

w/ $N_i / N_{i+1} \cong \mathbb{G}_m$ or $\mathbb{G}_a$.

<u>Prop:</u> If G is sm. conn. LAG $/ k = \bar{k}$, and G commutative then G is split solvable.

<u>Cor:</u> If G is solvable $/ k = \bar{k}$ then G is split-solvable.

<u>Theorem</u> (Borel fixed pt theorem)

If G acts on a proper variety X, G split-solvable and $X(k) \neq \emptyset$ then G fixes some $x \in X(k)$.

<u>Pf:</u> Induct on $\dim G$. $G = (e)$ ✓

if $\dim G = 1$ then $G \cong \mathbb{G}_a$ or $\mathbb{G}_m$.

Choose $x \in X(k)$ consider map $G \longrightarrow X$, $g \longmapsto gx$

by properness, as $G = \mathbb{G}_m$ or $\mathbb{G}_a \subseteq \mathbb{P}^1$

the corresp. rat'l map

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, dashed] & X \\ G \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}$$

extends to a morphism $\mathbb{P}^1 \to X$

"valuative criteria for properness"

action of G on itself extends (uniquely)

to an action of G on $\mathbb{P}^1$ (fixing ∞)

follows that $\mathbb{P}^1 \xrightarrow{\varphi} X$ respects the G-action.

$$g, \ell \longmapsto \varphi(g\ell)$$

$$G \times G \subseteq G \times \mathbb{P}^1 \rightrightarrows X$$

$$g, \ell \longmapsto g \cdot \varphi(\ell)$$

agree on $G \times G$ dense in $G \times \mathbb{P}^1$

so agree.

$\Rightarrow \infty$ fixed $\Rightarrow \varphi(\infty)$ fixed by G. ✓

Inductive case: let $N \triangleleft G$ w/ $G/N \cong \mathbb{G}_m$ or $\mathbb{G}_a$

i.e. N split solvable.

N acts on X, by induction, get $x \in X(k)$ N-fixed.

Consider the orbit of x, Gx.

$G \to X$
$g \mapsto gx$

points$/\bar{k}$ $\Big\{$ if $y = gx$ $\quad n \cdot y = ngx = gg^{-1}ngx = g\underbrace{(g^{-1}ng)}_{\in N}x$

$= gx = y$

$\Rightarrow N$ acts trivially on Gx

$\Rightarrow N$ acts trivially on $\overline{Gx} = Y \subseteq X$

get an action of G/N on Y, proper $\Rightarrow$ fixed pt.

$\Rightarrow$ fixed point for g. $\square$

<u>Cor</u> (Lie-Kolchin)

For G split-solvable, let $\rho: G \to GL(V)$ any rep. then $\exists$ basis for V

s.t. $\rho(g)$ is upper Δ all $g \in G(\bar{k})$.

<u>Pf</u>: let $Fl(V)$ be the variety of full flags of subspaces in V.

$Fl(V)(k) = \{ W_0 \subseteq \dots \subseteq W_n = V \mid \dim W_i = i \}$

this is a <u>projective variety</u>. G acts on $Fl(V)$ by its action on V,

and so Borel $\Rightarrow \exists$ fixed point. $W_0 \subseteq \dots \subseteq W_n$

Choose basis w/ $W_i = \langle e_1, \dots, e_i \rangle$ we find $g \in G(\bar{k})$

$$\rho(g) = \begin{bmatrix} * & * & & * \\ 0 & * & & \\ & & \ddots & \\ 0 & 0 & & * \end{bmatrix} \quad \square.$$

<u>Cor</u>: If G is split solvable then $G \cong U \rtimes T$ T torus, U unipotent

Pf: U closed defined by diag chars $= 1$.

or by $(g-1)^n = 0$ $\Rightarrow U \triangleleft G$ G/U comm. sep.

$g = g_s g_u$ Set $T = G \cap \text{diag. in } GL_n$

$T \simeq G/U$.

i.e. $G \to G/U$ admits a section via T.]].

above $\quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}$