temp

Def Let G be a smooth LAG over k.
we say a closed subgp $U \leq G$ is "u-good"
if U is unipotent, normal, smooth, connected.

Lemma (21.4.1 Conrad) Let G be a smooth LAG$/k$
then $\exists$ a (unique) u-good subgp containing all other u-good subgps.
This is called the unipotent radical $R_{u,k}(G)$ or $R_u(G)$

Def G is reductive if $R_{u,\bar{k}}(G_{\bar{k}}) = (e)$
G is pseudo-reductive if $R_{u,k}(G) = (e)$

Fact: G pseudoreductive over k perfect $\iff$ G reductive.

Remark: if G is split solvable then from before $G \simeq U \rtimes T$
U = all unipotent elements (cut out by $(T-1)^n = 0$ in $\mathrm{End}(V)$)
$\Rightarrow U = R_u G$

visualization:

radical

$$\left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right]$$

Pf main idea:

if $U_1, U_2 \trianglelefteq G$ are u-gond then $\exists$ U u-gond containing both,

$U_1 \trianglelefteq G \rightsquigarrow U_2$ acts on U_1 by conjugation.

$$U_1 \rtimes U_2 \longrightarrow G$$

Let U image, U closed sm...

$$\boxed{\begin{array}{l}\text{Last time: if } N \trianglelefteq G \text{, then} \\ \quad 1)\ N, G/N \text{ unipotent} \Rightarrow G \text{ unipotent.} \\ \quad 2)\ \text{if } G \text{ unipotent} \Rightarrow N \text{ \& } G/N \text{ unipotent}\end{array}}$$

$\Rightarrow$ (U_1, U_2 unipotent $\Rightarrow U_1 \rtimes U_2$ unipotent (by 1))

$U_1 \rtimes U_2$ unipotent $\Rightarrow U$ unipotent since U is a quotient.

$\square$.

Towards the structure thy:

<u>Sketch</u> (simplifying assumption $k = \bar{k}$)

G LAG sm $/k$ $\rightsquigarrow R_uG \trianglelefteq G$

$G^{red} \equiv G/R_uG$

replace $G = G^{red}$

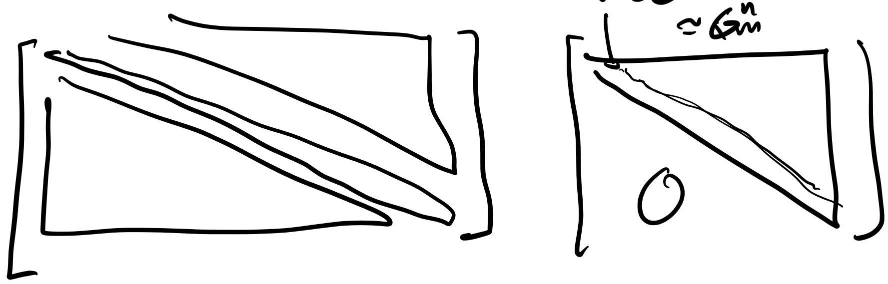
$$DG \to G \to G/DG \simeq \mathbb{G}_m^n$$

$\uparrow$ reductive, comm. $\Rightarrow$ torus

DG "semisimple" gp

G reductive $\qquad DG$ s.simple

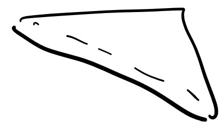
$T \subseteq G$ max'l torus $\simeq \mathbb{G}_m^n$



B max'l solvable subgp. "Borel"

$\cong$

$U \rtimes T$



Torus acts on U.

U generated by a bunch of copies of $\mathbb{G}_a$

$$\begin{array}{cc} i \to & \begin{bmatrix} 1 & 0 & * & 0 \\ & \ddots & & 0 \\ 0 & & & 1 \end{bmatrix} \\ & \qquad\quad \uparrow j \end{array}$$

single slot.

$$\simeq \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \simeq \mathbb{G}_a$$

embeds $\mathbb{G}_a \to B \subseteq G$

normalized by T.

for each such special embedding $\mathbb{G}_a \subseteq B \subseteq G$

$\circlearrowleft$ $\circlearrowright$ conjugation action.
T

given by list of $\mathbb{Z}$'s

$\mathbb{G}_m^n \circlearrowright \mathbb{G}_a$ $\qquad \mathrm{Aut}(\mathbb{G}_a) \subseteq \mathrm{Aut}(\mathbb{P}^1) \simeq \mathbb{G}_m$

fix ∞ & 0

$\mathbb{G}_m \longrightarrow \mathrm{Aut}(\mathbb{G}_a) = \mathbb{G}_m$

mult. by scalars.

$\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$

Fact: $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$ via

$(\lambda \mapsto \lambda^i) \leftarrow i$

For each $\mathbb{G}_a \to B \subseteq G$ normalized, get an action

given by $\mathrm{Hom}(T, \mathrm{Aut}\, \mathbb{G}_a)$

$$\begin{aligned} &\mathrm{Hom}(T, \mathrm{Aut}\, \mathbb{G}_a) \\ &= \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m) = \prod^n \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \\ &= \mathbb{Z}^n \end{aligned}$$

Def $\mathrm{Hom}(T, \mathbb{G}_m) = \chi_T$

$\simeq \mathbb{Z}^n$

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} 1 & & e_{ij} \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{bmatrix}$$

$$\begin{pmatrix} 1 & & \lambda_i \lambda_j^{-1} e_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}$$

char assoc. to e_{ij} ($\mathbb{G}_a \to GL_n$)

$\mathbb{G}_m^n \longrightarrow \mathbb{G}_m$

$\lambda_1, \dots, \lambda_n \longrightarrow \lambda_i \lambda_j^{-1}$

span of characters

$\updownarrow$

kernel of $\mathbb{Z}^n \xrightarrow{\Sigma} \mathbb{Z}$

"shadow of semisimple

SL_n PGL_n

G

DG $G/Z(G)$

Thm: If G smooth connected LAG $/ k = \bar{k}$

let B = a solvable smooth connected subgp

then B is maximal $\iff$ G/B proper variety.

& all such B are conjugate by elts of $G(k)$.

Quotients

Def let $G \circlearrowright X$ gp scheme act on scheme X.

we say that Y is a categorical quotient of X by G

if Y is the coequalizer of

$$\left[\; G \times X \rightrightarrows X \;\right] \longrightarrow Y$$

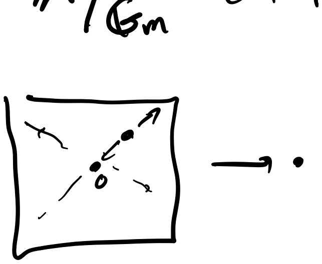
$$g, x \longmapsto gx \qquad\qquad g, x \longmapsto x$$

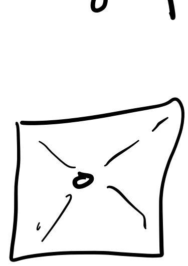
"best quotient candidate in cat of schemes"

Warning: can behave badly

$\mathbb{A}^2 / \mathbb{G}_m$ cat. quotient is a single point.

$\mathbb{A}^2 \setminus \{0\} / \mathbb{G}_m \cong \mathbb{P}^1$





Want: "good geometric quotients"

Focus on case $H \leq G$ closed subgp of a sm. LAG/k

Want cat. quotient G/H to have some extra nice properties:

Def X is a coset space for H in G (we write $X = G/H$)

if we have morphism $G \xrightarrow{\pi} X$, π flat, surjective.

s.t. $G \times H \xrightarrow{gh} G \to X$, $G \times H \xrightarrow{(g,h)\mapsto g} G \to X$ commutes.

s.t. the map $G \times H \longrightarrow G \times_X G$

$(g, h) \longmapsto (g, gh)$ is a isom.

unpack this:

$G \xrightarrow{\pi} X$ $(= G/H)$ surjective

if a pair of points g_1, g_2 map to same pt in X, they differ by H on right.

$\{$pairs of pts that map to same in $X\} = G \times_X G$

if know $X = G/H$ then if g_1, g_2 same image,

could take difference $g_1 g_2^{-1} = h \in H$

$G \times_X G \overset{\text{secretly}}{=} \{(g_1, g_2) \mid g_1 H = g_2 H\}$

$= \{(g_1, g_2) \mid g_2 = g_1 h \text{ some } h \in H\}$

$= \{(g_1, g_1 h) \mid h \in H\} \leftrightarrow \underbrace{\{g_1 \in G, h \in H\}}_{G \times H}$

Prop: If $H \leq G$ closed sm. subgp of G sm. LAG$/k$

then $\exists$ a cont space $X = G/H$.