Existence of quotients

Thm (18.1.7 in Conrad's notes)

Let G be a smooth LAG$/k$ and $H \subseteq G$ closed.

Then $\underset{\substack{\shortparallel \\ X}}{G/H}$ exists as a quasiprojective coset space, and $X(\bar{k}) = G(\bar{k})/H(\bar{k})$

Pf (sketch)

example: $G = GL(V)$ then G acts on V & $\mathbb{P}(V)$

Let $H = \mathrm{Stab}([L])$ $\quad [L] \in \mathbb{P}(V)(k)$ i.e. $L \subseteq V$ a line

i.e. $H = \left[\begin{array}{c|c} * & \\ 0 & * \\ \vdots & \\ 0 & \end{array}\right]$ $\quad (L = e_1 k)$

then $X = \mathbb{P}(V)$ works. i.e. $G \circlearrowright \mathbb{P}(V)$

$$G \to \mathbb{P}(V)$$
$$g \mapsto g[L] = [gL]$$

$g, g' \longmapsto$ same $[L']$ iff $g[L] = g'[L] = [L']$

$$G \times_X G \xleftarrow{\sim} G \times H \iff g^{-1}g'[L] = [L] \quad g^{-1}g' \in H$$
$$(g, gh) \longleftrightarrow g, h \qquad g' = gh \text{ some } h \ldots$$

Recall: all gps can be idented as stab. of lines

Choose $G \hookrightarrow GL(V)$ where H is the stabilizer of some line $[L] \in \mathbb{P}(V)$, consider X = orbit of $[L]$ under G.

i.e. $G \longrightarrow \mathbb{P}(V)$
$g \longmapsto g\cdot[L] = [gL]$

"Closed orbit lemma": orbits under smooth conn. gp actions are locally closed, min'l dim'l orbits are closed and orbits are smooth w/ ind., red. scheme structure

X is smooth, q. projective variety $\subseteq \mathbb{P}(V)$

$G \xrightarrow{\pi} X$

consider $\bar{k}$ points, we see

$G \times H \longrightarrow G \times_X G$ is bijective.
$g, h \longrightarrow g, gh$

$g \xrightarrow{\pi} g[L]$
$gh \longrightarrow gh[L]$
$= g[L]$

inverse? $g, g' \in G \times_X G$ i.e. $\pi g' = \pi g$
i.e. $g'L = gL$
then $g^{-1}g'L = L \Rightarrow g^{-1}g' \in H$
$g' = gh$ some $h \in H$ (h unique)

$g, g' \mapsto g, h$ inverse. $\Longleftarrow$

also shows fibers $G \xrightarrow{\pi} X$ are cosets $gH \simeq H$ as varieties.

so fibers are all same dimension, G, X smooth

$\Rightarrow$ "miracle flatness" π flat. $\square$

<u>Remark</u>:

if $H \subseteq G$ as above, can consider the k-space $\mathrm{Pre}(G/H)$

$$R \longmapsto G(R)/H(R)$$

this is rarely representable by a scheme.

But G/H as above is the sheafification in étale topology

<u>ex</u>: if k^s/k is the separable closure of k

w/ Gal gp Γ then

$$G/H(k) = \left(G(k^s)/H(k^s)\right)^{\Gamma}$$

<u>observe</u>: if X is a variety quasi-projective. over k then

$$X(k^s) \circlearrowleft \Gamma \qquad X(k) = X(k^s)^{\Gamma}$$

G smooth connected LAG$/k = \bar{k}$

<u>Def</u> a subgp $B \subseteq G$ is called sol. sm con.

if it is solvable, smooth & connected.

Def $B \subseteq G$ is **Borel** if it is a max'l solvable conn subgp.

(**Def** $B \subseteq G/_{k \neq \bar{k}}$ is Borel if $B_{\bar{k}} \subseteq G_{\bar{k}}$ is Borel)

Lemma: If $B \leq G$ is solvable conn subgp of max'l dim

then G/B is proper.

Pf: Choose $G \hookrightarrow GL(V)$ s.t. $B = \underset{= N_G(L)}{\text{normalizer of a line}}$ $L \subseteq V$.

consider the action of G on $Fl(V/L)$

Since $k = \bar{k}$ B is split solvable, so Borel fixed pt

theorem $\Rightarrow$ $B \circlearrowright Fl(V/L)$ fixes some point

$\overline{W}_2 \subseteq \overline{W}_3 \subseteq \dots \subseteq \overline{W}_n = V/L$ subspaces of dim $\overline{W}_i$ is $i-1$

$\Rightarrow$ if $W_i/L = \overline{W}_i$, $W_1 = L$ then B fixes

$\underbrace{W_1 \subseteq \dots \subseteq W_n = V}_{\mathcal{F} \in Fl(V)}$ in $Fl(V)$. (w/r/t action $GL(V)$)

$\mathrm{Stab}_G(\mathcal{F})$

note: $B = N_G(\mathcal{F})$ since $B \subseteq N_G(\mathcal{F}) \subseteq N_G(L) = B$

consider orbit $G \longrightarrow Fl(V)$

$g \longmapsto g\mathcal{F}$

stab'zer of $\mathcal{F}$ is B $\underset{\text{via prev logic}}{\rightsquigarrow}$ orbit of $\mathcal{F} = G/B$

By "closed orbit lemma", suffices to show that orbit has min'l dim among all possible G-orbits

Sketch: if $G\mathcal{F}'$ another orbit, then $G\mathcal{F}' \cong G/N_G(\mathcal{F}')$

and $N_G(\mathcal{F}') = B'$ is upper Δ-izable in rep $\Rightarrow$ solvable.

$\Rightarrow \dim B' \leq \dim B \Rightarrow \dim G/B' \geq \dim G/B$

$$\underset{G\mathcal{F}'}{\overset{\|}{G/B'}} \qquad \underset{G\mathcal{F}}{\overset{\|}{G/B}}$$

$\Rightarrow \dim G\mathcal{F}$ is minimal among orbits

$\Rightarrow G\mathcal{F}$ closed in $\underset{\text{projective}}{Fl(V)}$

$\quad\; \| \; G/B$

$\Rightarrow G/B$ projective variety!

Thm (22.1.1 ish)

All Borel subgps $B \leq G$ (G sm. connected LAG $/ k=\bar{k}$) are conjugate.

i.e. $B, B' \subseteq G$ Borel $\Rightarrow \exists\, g \in G(k)$ w/ $gBg^{-1} = B'$

Pf: if B, B' as above, w/ B max'l dim.

consider action of B' on $\underset{\text{projective.}}{G/B}$ (via the G action restricted)

Borel fixed pt $\Rightarrow$ B' fixes some coset gB

But stab of gB is gBg^{-1} $\Rightarrow$ $B' \subseteq gBg^{-1}$

But B' max'l $\Rightarrow$ $B' = gBg^{-1}$. $\square$

<u>Cor</u> (22.1.4) if G is smooth connected LAG$/k = \bar{k}$ & G is not unipotent, then G contains a copy of $\mathbb{G}_m$.

<u>Pf</u>: Let $B \subseteq G$ be Borel, B split soluble $\Rightarrow$ $B \cong U \rtimes T$

U unipotent, T torus. we need to show $T \neq (e)$.

If $B = U$, consider G/B ($\neq$ pt since G not unipotent)

choose $G \to GL(V)$ s.t. B = norm. of a line $L \subseteq V$

get a map $B \to \mathrm{Aut}(L) \cong \mathbb{G}_m$ B unipotent so this can't happen

$\cup$

$\mathbb{G}_a$ normal in B

so B centralizes L, so get B = Centralizer of $v \in L \setminus \{0\}$

consider orbit of v under G

$$\begin{tikzcd} G \arrow[r] \arrow[d] & V = \mathbb{A}(V) \\ G/B \arrow[ur] & \end{tikzcd} \qquad g \mapsto gv$$

G/B projective smooth not a point   so $G/B \to V$ constant.

$\Rightarrow B = G$. ↯