Cor (of projectivity of G/B)

If G is smooth connected LAG $/ k = \bar{k}$ and every $g \in G(k)$ is semisimple then G is a torus

Pf: Let $B \leq G$ be Borel and write $B = \mathcal{U} \rtimes T$

but because every $g \in G(k)$ is unipotent $\Rightarrow \mathcal{U}(k) = (e)$
$\Rightarrow \mathcal{U} = (e)$

Claim: $B = T \subseteq Z(G)$

choose $b \in B(k)$

$$\begin{tikzcd} G \arrow[r] \arrow[d, bend right] & G \\ G/B \arrow[ur, bend right] & \end{tikzcd} \qquad g \longmapsto gbg^{-1}$$

claim: map factors like this

$g' = gb'$

$g'bg'^{-1} = gbg^{-1}$

$\parallel$

$gb'bb'^{-1}g^{-1}$

$B = T \Rightarrow B$ comm $\Rightarrow$ $\parallel$ gbg^{-1} ✓.

so we get factorization:

$$\begin{tikzcd} G \arrow[r] \arrow[d, bend right] & G \\ G/B \arrow[ur, bend right] & \end{tikzcd} \qquad g \longmapsto gbg^{-1}$$

but G/B projective, G affine

$\Rightarrow$ image of G/B in G is a point.

$b =$ image of $ebe^{-1} \Rightarrow \text{im} = b$. □

example of nonsm. gp

$R_{L/k}\mathbb{G}_m$ $\quad k = \mathrm{char}\ p \quad L = k(\alpha) \quad \alpha^p = a \in k^* \setminus (k^*)^p$

$R_{L/k}\mathbb{G}_m(k) = L^* \qquad R_{L/k}\mathbb{G}_m(E) = (L \otimes E)^*$

$L = k \oplus k\alpha \oplus k\alpha^2 \oplus \cdots \oplus k\alpha^{p-1} = \mathbb{A}^p_k$

$b = \sum b_i \alpha^i \Longleftrightarrow (b_0, \dots, b_{p-1})$

$b \rightsquigarrow$ matrix describing mult by b as a lin trans on $L = M_b$

$1 \longmapsto \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{M_1} \qquad \alpha \longmapsto \underbrace{\begin{bmatrix} 0 & 0 & & a \\ 1 & 0 & & 0 \\ & 1 & & \vdots \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix}}_{M_\alpha} \qquad \alpha^p = a$

$\det M_b \neq 0$ defines an open subscheme of $\mathbb{A}^p_k$

which we call $R_{L/k}\mathbb{G}_m$

inside of $R_{L/k}\mathbb{G}_m$ can consider $R_{L/k}\mathbb{G}_m[p]$

closed subgp defined by

$b^p = 1$

Actually just consider $\mathbb{G}_m[p]$ $\quad x^p - 1$

over k char p $\quad \uparrow$ not smooth.

$$\mathbb{G}_m[p](R) = \{ r \in R^* \mid r^p = 1 \}$$

over k, $x^p - 1 = (x-1)^p$ only 1 as a root.

$SO_n \quad O_n^+ \quad$ char 2

not smooth.

Def: A parabolic subgp of G (sm. conn. LAG/k) is a sm. subgp s.t. G/P is proper.

Note: Borels are parabolic, but nonnormal parabolics need not exist in general. ($G = P$ is parabolic)

ex: $\mathbb{H}/\mathbb{R}$ quaternions $\quad GL_1(\mathbb{H})$

$$GL_1(\mathbb{H})(\mathbb{R}) = \mathbb{H}^*$$

$$GL_1(\mathbb{H})_{\mathbb{C}} \simeq GL_1(\mathbb{H}_{\mathbb{C}}) = GL_1(M_2(\mathbb{C})) = GL_2(\mathbb{C})$$

$GL_2(\mathbb{R})$ $\quad GL_2(\mathbb{C})$ have Borel subgps $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

But $GL_1(\mathbb{H})$ doesn't have such a Borel.

Def: A smooth connected LAG $/k$ is called quasisplit if it has a Borel subgp (defined over k!)

(M. Henle: modern geometries, noneuclidean, projective & discrete)

Prop $P \subseteq G$ smooth connected LAG $/k$. P is parabolic iff $P_{\bar{k}}$ contains a Borel of $G_{\bar{k}}$.

Pf: WLOG, can assume $k = \bar{k}$.

if $B \subseteq P$ B Borel WTS G/P projective

consider $G/B \twoheadrightarrow G/P$ (quotients exist)

$\Rightarrow G/P$ proper ✓

Conversely, suppose G/P projective. Consider action of B on cosets G/P by left mult.

Borel fixed pt theorem $\Rightarrow \exists$ fixed point gP under B

$\Rightarrow gP = BgP \Rightarrow P = g^{-1}BgP \Rightarrow g^{-1}Bg \subseteq P$

But $g^{-1}Bg$ is also Borel so P contains a Borel ⊠.

Theorem: (Chevalley)

If G sm. connected LAG$/k$, $P \subseteq G$ parabolic then P is connected and $P = N_G(P)$.

Theorem: (Grothendieck)

If G is sm. connected LAG$/k$, $\exists$ a torus $T \subseteq G$ s.t. $T_{\bar{k}} \subseteq G_{\bar{k}}$ is a maximal torus

Will show: if $B \subseteq G$ is Borel, so $B \cong U \rtimes T$ then $T_{\bar{k}}$ is a maximal torus of $G_{\bar{k}}$

Theorem: If G is sm. connected LAG$/k = \bar{k}$ then all maximal tori of G are conjugate.

Remark: If $B, B' \subseteq G_{/k=\bar{k}}$ Borel then $B' = gBg^{-1}$ some $g \in G(k)$

Observation: if $T \subseteq G_{/k=\bar{k}}$ a torus, then T is solvable, sm. connected.

$\Rightarrow T$ contained in a max'l solv. conn. i.e. a Borel.

So if T max'l then $T \subseteq B$ Borel

an ottrhd if $B = U \rtimes T$ and T' some oth max'l torus then

$T' \subseteq B'$ B' conj. to B

So after conjugating, $gT'g^{-1}, T \subseteq B$

So, being max'l in Borel = max'l in group.