Abstract nonsense "spaces" | Actual AGeom | Alg Groups

Last time:

group spaces, affine group schemes, Hopf algebras.

Group space:

functor G: k-algebras $\to$ groups

example: $\mathbb{G}_m(R) = R^\times$

observation: compose w/ forgetful functor gps $\to$ sets

result is representable.

warm-up: $\mathbb{A}^1(R) = R$

representable by A $\quad \mathrm{Hom}_{k\text{-alg}}(A, R) = R$

$$\mathrm{Hom}_{k\text{-alg}}(A,R) = \mathbb{A}^1(R)$$

$A = k[x] \qquad \mathrm{Hom}_k(k[x], R) = R$

$x \mapsto a \qquad a$

$$\mathbb{G}_m(R) = R^\times = \mathrm{Hom}_k(A, R) = \mathrm{Hom}_k(k[x,x^{-1}], R)$$

$A = k[x, x^{-1}] \qquad x \longmapsto r \in R^\times$

$$\mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$$

$$\mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

$$\mathbb{G}_m(R) \times \mathbb{G}_m(R) \longrightarrow \mathbb{G}_m(R)$$

$$\operatorname{Spec} k[x,x^{-1}] \times \operatorname{Spec} k[x,x^{-1}] \longrightarrow \operatorname{Spec} k[x,x^{-1}]$$

$$\operatorname{Spec} k[x,x^{-1}] \otimes k[x,x^{-1}] \longrightarrow \operatorname{Spec} k[x,x^{-1}]$$

$$\left(k[x,y,x^{-1},y^{-1}] \simeq \right) k[x,x^{-1}] \otimes k[x,x^{-1}] \longleftarrow k[x,x^{-1}]$$

comultiplication

$$\mathbb{G}_m(R) \times \mathbb{G}_m(R) \longrightarrow \mathbb{G}_m(R)$$

$$\underset{\underset{R^\times}{\cap}}{a}, \ \underset{\underset{R^\times}{\cap}}{b} \longmapsto ab$$

$$\overset{a}{\mathbb{G}_m(R)} \times \overset{b}{\mathbb{G}_m(R)} \left(\operatorname{Spec} k[x,x^{-1}] \otimes_k k[x,x^{-1}] \right)(R)$$

$$\| \qquad\qquad = \qquad\qquad \| \quad x\otimes 1 \mapsto a, \ 1\otimes x \mapsto b$$

$$(\mathbb{G}_m \times \mathbb{G}_m)(R) \qquad \operatorname{Hom}(k[x,x^{-1}] \otimes k[x,x^{-1}], R)$$

$$\|$$

$$a \in \operatorname{Hom}(k[x,x^{-1}], R) = R^\times$$

$$\times$$

$$b \in \operatorname{Hom}(k[x,x^{-1}], R) = R^\times$$

$$x \longmapsto x \otimes x$$

$$k[x,x^{-1}] \longrightarrow k[x,x^{-1}] \otimes k[x,x^{-1}] \longrightarrow R$$

$$x \mapsto ab \qquad x\otimes 1 \mapsto a \quad 1 \otimes x \mapsto b$$

comultiplication: $\mathbb{G}_m \leftarrow \mathbb{G}_m \times \mathbb{G}_m$

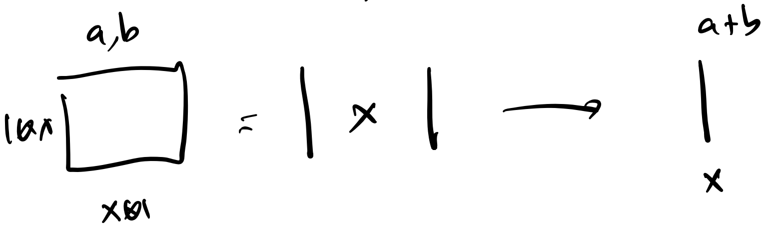
$$k[x,x^{-1}] \longrightarrow k[x,x^{-1}] \otimes k[x,x^{-1}]$$
$$x \longmapsto x \otimes x$$

defines product in $\mathbb{G}_m$.

Exercise (now): do same for $\mathbb{G}_a(R) = (R,+)$

Guess: $x \mapsto x\otimes 1 - 1 \otimes x$ $\quad \mathbb{G}_a = \operatorname{Spec} k[x]$

$x \mapsto x \otimes 1 + 1 \otimes x$



value of x coord in image $= a+b = x$
comes from sum of coords in domain a, b
$x\otimes 1 + 1 \otimes x$

~~Example?~~ exercise

$R \mapsto (R[[t]], +) = \mathbb{A}^\infty = \mathbb{A}^{\mathbb{N}}$

$\mapsto (R((t)) \equiv \{\sum_{i=d}^{\infty} a_i t^i \mid a_i \in R\}, +)$

which of these are representable?

$\mapsto (R((t))^\times, \cdot)$

$\mapsto (R[[t]]^\times, \cdot)$

sub exercise: $\sum_{i=0}^{\infty} a_i t^i$ invertible mult. $\Leftrightarrow a_0 \in R^\times$

set theoretically

$\mathbb{G}_m(R) \subset \mathbb{G}_a(R)$

$R^\times \subset R$

<u>more examples</u>

$$(M_n, +) = \mathbb{A}^{n^2}, + = \mathrm{Spec}\, k[x_{ij} \mid i,j \in \{1,\dots,n\}]$$

$$\underbrace{\mathbb{G}_a \times \cdots \times \mathbb{G}_a}_{n^2 \text{ times}} = \mathbb{G}_a^{n^2}$$

$$(GL_n, \cdot) = k[x_{ij} \mid i,j \in \{1,\dots,n\}][\det[x_{ij}]^{-1}]$$

$$\downarrow$$

$$R$$

$$SL_n(R) = \{T \in M_n(R) \mid \det T = 1\}, \cdot$$

$$O_n(R) = \{T \in M_n(R) \mid T^t T = 1\}$$

$$Sp_{2n}(R) = \{T \in M_{2n}(R) \mid T^t \Omega T = \Omega\}$$

$$\Omega = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$$

$$PGL_n(R) = \{T \in GL_n(R)\} / \lambda T \sim T \text{ if } \lambda \in R^\times$$

$$\text{"}PSL_n\text{"}(R) = \{T \in SL_n(R)\} / \lambda T \sim T \text{ if } \lambda \in R^\times$$

$\lambda \in \mu_n(R)$

$$\mu_n(R) = \{a \in R \mid a^n = 1\}$$

μ_n subgrp of $\mathbb{G}_m$

" $PSL_n(\mathbb{R})$ " $< PGL_n(\mathbb{R})$

$$\det \begin{bmatrix} 2\lambda & 0 \\ 0 & \lambda \end{bmatrix} = 2\lambda^2 \qquad \overline{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}} \in PGL_2(\mathbb{Q})$$

<u>Abstract nonsense</u> Fiber products

<u>Def</u> X, Y, S sets, $f: X \to S$, $g: Y \to S$

define $X \times_S Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$

<u>univ. property</u>:

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

<u>Def</u> X, Y, S spaces & morphisms $f: X \to S$, $g: Y \to S$

define $(X \times_S Y)(R) = X(R) \times_{S(R)} Y(R)$

<u>Claim</u>: if W space,

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y & S
\end{tikzcd}$$

$W \to X \times_S Y$

$$\begin{tikzcd}
W(R) \arrow[r, dashed, "?"] \arrow[rr, bend left] \arrow[dr, bend right] & X(R) \times_{S(R)} Y(R) \arrow[r] \arrow[d] & X(R) \arrow[d] \\
& Y(R) \arrow[r] & S(R)
\end{tikzcd}$$

if $\exists$ nat trans $\alpha : W \to X \times_S Y$

s.t.

$$\begin{tikzcd}
W \arrow[r] \arrow[rr, bend left] \arrow[dr, bend right] & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

commutes

then by uniqueness in diagram

$$\begin{tikzcd}
W(R) \arrow[r, "!"] \arrow[rr, bend left] \arrow[dr, bend right] & X(R) \times_S Y(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{fib. prod}"] & X(R) \arrow[d] \\
& Y(R) \arrow[r] & S(R)
\end{tikzcd}$$

$R \to R'$

$$\begin{tikzcd}
W(R) \arrow[r] \arrow[d] & X(R) \times_{S(R)} Y(R) \arrow[r] \arrow[d] & Y(R) \arrow[d] \\
W(R') \arrow[r] & X(R') \times_{S(R')} Y(R') \arrow[r] \arrow[d] & Y(R') \arrow[d] \\
& {} & {}
\end{tikzcd}$$

$$\tilde{X}(R') \longrightarrow S(R')$$

<u>Def</u> if S a space, an S-space is a space X w/ morphism to S.

$$X \to S$$

<u>Observe</u>: fiber products $\_\times_S X$ give a functor from S-spaces to X-spaces

$$Y \to S \quad \rightsquigarrow \quad Y \times_S X \longrightarrow X$$

$$\begin{tikzcd} Y' \arrow[d, "f"'] \arrow[r] & S \\ Y \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} Y' \times_S X \arrow[d, "f \times_S X"'] \arrow[r] & X \\ Y \times_S X \arrow[ur] & \end{tikzcd}$$

<u>Def</u> if $X \xrightarrow{f} S$ is a morphism of spaces, we say f is (affine) representable if for every $s \in S(R) = \mathrm{Hom}(\mathrm{Spec}\, R, S)$

$$\begin{tikzcd} & X \arrow[r] & S \\ X \times_S \mathrm{Spec}\, R \arrow[ur] \arrow[r] & \mathrm{Spec}\, R \arrow[ur] & \end{tikzcd}$$

$X \times_S \mathrm{Spec}\, R$ is representable by a ring

i.e. $X \times \mathrm{Spec}\, R \simeq \mathrm{Spec}\, A$

<u>Def</u> $f: X \to Y$ is open if for all $y \in Y(R)$

$X \times_Y \operatorname{Spec} R \to \operatorname{Spec} R$ is an open inclusion.

i.e. $X \times_Y \operatorname{Spec} R$ is a subfunctor of $\operatorname{Spec} R$ corresponds to $U(I)$ I ideal in R

similarly, closed morphisms.

<u>Rem:</u> closed morphisms are always representable (affine)
not open morphisms.

$\mathbb{A}^2 \setminus \{0\} \subset \mathbb{A}^2$