Gluing (Quotients are ~~hext~~ the future.)

Problem: k-spaces don't glue "correctly"

Def: If
$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$
is a diagram of sets the pushout $X \sqcup_S Y$

$X \sqcup_S Y$

is defined as $X \cup_S Y := X \sqcup Y / S$

$$= \{[x;1], [1;y]\} \Big/ {\scriptstyle [f(s);1] \sim [1;g(s)]}$$

Similar nature:

Def $S \underset{g}{\overset{f}{\rightrightarrows}} X$ maps of sets the coequalizer

$$\mathrm{coeq}(f,g) = X \big/ {\scriptstyle f(s) \sim g(s)}$$

Rem: $X \sqcup_S Y = \mathrm{coeq}(S \rightrightarrows X \sqcup Y)$

These have universal properties, categorical generalizations

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd} \quad \text{in } \mathcal{C} \qquad \begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \sqcup_S Y \end{tikzcd}$$

s.t. if
$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \qquad \exists! \ Z \to X \sqcup_S Y$$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d] \arrow[rd, bend left] & \\ Y \arrow[r] \arrow[rrd, bend right] & X \sqcup_S Y \arrow[rd, dashed] & {} \\ & & Z \end{tikzcd}$$

<u>Def/</u> if $\begin{tikzcd} S \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$ is a diagram of
lem
k-spaces

pushout $X \sqcup_S Y(R)$
$$\| $$
$$X(R) \sqcup_{S(R)} Y(R)$$

<u>ex:</u> $\tilde{\mathbb{P}}^1 = \mathbb{A}^1 \sqcup_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$

$$\operatorname{Spec} k[x] \sqcup^{\text{space}}_{\operatorname{Spec} k[x,x^{-1}]} \operatorname{Spec} k[x^{-1}]$$

$$\tilde{\mathbb{P}}^1(R) = \mathbb{A}^1(R) \amalg_{\mathbb{A}^1\setminus\{0\}(R)} \mathbb{A}^1(R)$$

$$\begin{tikzcd} \mathbb{A}^1\setminus\{0\}(R) = R^* \arrow[r] \arrow[d] & \mathbb{A}^1(R) \arrow[d] \\ \mathbb{A}^1(R) \arrow[r] & \{[a;1] \cup [1;b]\} / [a;1] \sim [1;b] \text{ if } a = b^{-1} \end{tikzcd}$$

$x^{-1} \mapsto a^{-1}$, $x \mapsto a \in R$, $x \mapsto a$, $x^{-1} \mapsto a^{-1}$

<u>problem</u>: this isn't right

$\mathbb{P}^1_{\mathbb{Z}}(\mathbb{Z}) \ni [2;3]$

$\mathbb{P}^1(k[t]) \ni [t; t-1]$

<u>Ex</u>: characterize $\mathrm{Hom}_{k\text{schemes}}(\mathbb{A}^1, \mathbb{P}^1) = \mathbb{P}^1(k[t])$

Hint/red herring? our $\mathbb{Z}[\sqrt{-5}]$ points of proj space aren't always given by pairs of elements.

<u>Compare</u>: in k-affine schemes? $= (k\text{-alg})^{op}$

$$\begin{tikzcd} \operatorname{Spec} k[x,x^{-1}] \arrow[r] \arrow[d] & \operatorname{Spec} k[x] \\ \operatorname{Spec} k[x^{-1}] & \end{tikzcd}$$

$$\begin{array}{ccc} k[x,x^{-1}] & \longleftarrow & k[x] \\ \uparrow & & \uparrow \\ k[x^{-1}] & \longleftarrow & k[x] \times_{k[x,x^{-1}]} k[x^{-1}] \end{array}$$

$= \{ (f(x), g(x^{-1})) \mid f(x) = g(x^{-1}) \}$

$= k$ (also not cord ring corresp. to $\mathbb{P}^1$)

$\operatorname{Spec} k \neq \mathbb{P}^1$

(not too bad - global fcns on $\mathbb{P}^1$ are constant)

$$\begin{array}{ccc} \overset{?}{\boxed{\mathbb{A}^1}} & \longleftarrow & \mathbb{A}^1 \setminus \{0\} \\ \uparrow & & \uparrow \\ \mathbb{A}^1 \setminus \{1\} & \longleftarrow & \mathbb{A}^1 \setminus \{0,1\} \end{array} \qquad \begin{array}{ccc} k[x,x^{-1}] & \longleftarrow & ? \\ \downarrow & & \downarrow \\ k[x,x^{-1},(x-1)^{-1}] & \longleftarrow & k[x,(x-1)^{-1}] \end{array}$$

$? = \left\{ \left(\frac{f(x)}{x^n}, \frac{g(x)}{(x-1)^m} \right) \middle| \frac{f(x)}{x^n} = \frac{g(x)}{(x-1)^m} \right\}$

$= \{ (h(x), h(x)) \mid h(x) \in k[x] \} = k[x]$

if we did this in k-spaces instead of in k-affine schemes would go wrong

way to see this:

$$\mathrm{Hom}_{k\text{-spcs}}\left(\mathrm{Spec}\, k[x],\ \mathrm{Spec}\, k[x,x^{-1}] \amalg \mathrm{Spec}\, k[x,(x-1)^{-1}] \atop \mathrm{Spec}\, k[x,x^{-1},(x-1)^{-1}]\right)$$

$$= \mathrm{Hom}_{k\text{-spcs}}\left(\mathrm{Spec}\, k,\ \text{---} \text{''} \text{---}\right)$$

<u>This should be familiar</u>

Compare modules:

$$0 \to A \to B \to C \to 0 \qquad Y$$

$$0 \to \mathrm{Hom}(Y,A) \to \mathrm{Hom}(Y,B) \to \mathrm{Hom}(Y,C)$$

as <u>functors</u>

$$0 \to \mathrm{Hom}(-,A) \to \mathrm{Hom}(-,B) \to \mathrm{Hom}(-,C)$$

(ring A $\quad$ $\mathrm{Spec}\, A$) $\qquad (\mathrm{Spec}\, A)(R) = \mathrm{Hom}_{k\text{-spcs}}(\mathrm{Spec}\, R, \mathrm{Spec}\, A)$

$$A \longrightarrow \mathrm{Hom}(-,A)$$

mods $\longrightarrow$ functs (Modules, Ab) = Ab cat

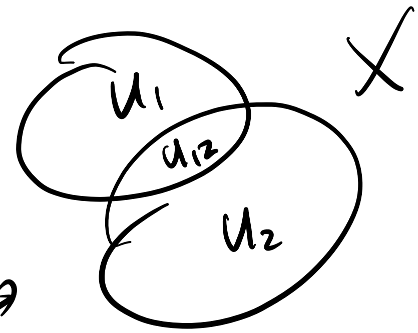
Ab cat $\qquad$ preserves l.exactness.

i.e. kernel preserving: $\quad \ker(B \xrightarrow{f} C) = \mathrm{eq}(B \underset{f}{\overset{0}{\rightrightarrows}} C)$

rel to fiber products

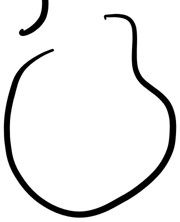
not coker preserving $\quad \mathrm{Coker}(A \xrightarrow{g} B) = \mathrm{coeq}(A \underset{g}{\overset{0}{\rightrightarrows}} B)$

Another perspective



$X = U_1 \cup_{U_{12}} U_2$

Y



$\xrightarrow{\varphi}$

$Y_1 = \varphi^{-1} U_1 \quad Y_2 = \varphi^{-1} U_2$

$Y_{12} = \varphi^{-1} U_{12}$

$Y = Y_1 \cup_{Y_{12}} Y_2 \qquad Y_\bullet \to X_\bullet$ compatible

neat way to define maps to glued stuff well

use topology of domain.

Fundamental idea: maps give sa form a sheaf!

$$Y = \bigcup V_i \qquad (f_i : V_i \to X)_i \rightrightarrows \begin{matrix} (f_i|_{V_i \cap V_j})_{ij} \\ (f_j|_{V_i \cap V_j}) \end{matrix}$$

$$\mathrm{Map}(Y, X) \to \prod_i \mathrm{Map}(V_i, X) \rightrightarrows \prod_{i,j} \mathrm{Map}(V_i \cap V_j, X)$$

equalizer = sheaf property

$\mathrm{Maps}(-, X)$ sheaf on Y.

Def A k-sheaf is a k-space X

s.t. for every R k-alg.

s.t. $\mathrm{Hom}_{k\text{-spe}}(-,X)$ restricted to open subspaces of Spec R in Zariski top

we get a sheaf.

i.e. if $f_1, \dots, f_r$ are jointly coprime

then $\mathrm{Spec}\, R_{f_i}$ give a Zariski cov. of Spec R

we're saying that

$$X(R) \to \prod X(R_{f_i}) \rightrightarrows \prod X(R_{f_i f_j})$$

$$\mathrm{Hom}(\mathrm{Spec}\, R, X) \to \prod \mathrm{Hom}(\mathrm{Spec}\, R_{f_i}, X) \rightrightarrows \prod \mathrm{Hom}(\mathrm{Spec}\, R_{f_i f_j}, X)$$

Claim: pushouts in k-sheaves are the "correct ones"

Def A k-scheme is a k-sheaf X w/ open subspaces

$U_i \simeq \mathrm{Spec}\, R_i$ and w/

$\coprod U_i \cap U_j \rightrightarrows \coprod U_i \to X$ is a coequalizer in kspes.

$\rightsquigarrow$ $|X|$ underlying top space from $\operatorname{Spec} R_i$

$\mathcal{O}_X$ structure sheaf.

OH today 11:30-12:30

$\operatorname{Spec} \mathbb{C}$ as an $\mathbb{R}$-scheme.

$$\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} = \operatorname{Spec} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$$
$$= \operatorname{Spec} \mathbb{C} \times \mathbb{C}$$

$X \rightsquigarrow \pi_0 X$