Office hours

generally: M 12:30-1:30

W 10-11

this week ~~W 10-11~~ → Tu 1-2

M: 12:30-1:30

Last time:

Def: a k-scheme is a k-sheaf X w/ open subspaces U_i w/ each $U_i \cong \operatorname{Spec} R_i$ and w/

$$\coprod_{ij} U_i \times_X U_j \rightrightarrows \coprod_i U_i \to X$$

"$U_i \cap U_j$"

is a coequalizer in $\boxed{k\text{-sheaves}}$ (not spaces!)

$$U_i(R) \cap U_j(R) = (U_i \times_X U_j)(R)$$

$$\begin{array}{ccccc} & & U_i & \to & X \\ & \nearrow & & \nearrow & \\ U_i \times_X \operatorname{Spec} A & \xrightarrow[\text{Zariski}]{\text{open}} & \operatorname{Spec} A & & \\ U_i \cap U_j & & R_j & & \\ & & U_j & & \end{array}$$

"schemes are affine k-spaces (sheaves) glued along open subspaces as k-sheaves"

HW: if $(X, \mathcal{O}_X)$ is a (loc.) ringed space

⇝ k-sheaf $R \to \operatorname{Hom}_{l.r.s}(\operatorname{Spec} R, (X, \mathcal{O}_X))$

"classical" Schemes $\hookrightarrow$ k-spaces

$$\xrightarrow[\sim]{} \text{k schemes (above def.)}$$

<u>Next bit</u>: some issues w/ working over fields not alg. closed

<u>Def</u> connected, irreducible, reducible, reduced

Zariski top connected

$X \neq X_1 \cup X_2$ properly, X_i closed

coord. rings no nilpotent elements.

<u>Def</u> A variety is an irred, reduced scheme of finite type <u>over a field</u>

locally $\cong$ Spec A

$A/_k$ f.g. k-alg.

<u>Def</u> X a k-scheme is geom. connected if $X_{\bar{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is connected

i.e. $X = \operatorname{Spec} A$ $\quad X_{\bar{k}} = \operatorname{Spec} A \otimes_k \bar{k}$

Similarly geom. reduced if $X_{\bar{k}}$ reduced

" irred if $X_{\bar{k}}$ irred.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1,0) \quad (0,1)$

Def Spec A is connected if $A \not\cong B \times C$ as k-algs.

i.e. A has no nontrivial idempotents

$e = e^2 \quad A = eA \times (1-e)A$

Def Spec A is reduced if A has no nilpotent elements

Def Spec A is integral if A is a domain

($\Leftrightarrow$ Spec A is reduced & irreducible)

ex: $\mathbb{C}/\mathbb{R}$ Spec $\mathbb{C}$ as an $\mathbb{R}$-scheme.

$\mathbb{C}$ domain $\Rightarrow$ Spec A integral $\qquad \mathbb{C} = \mathbb{R}[x]/_{x^2+1}$

$$\mathbb{C}_{\overline{\mathbb{R}}} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}[x]/_{x^2+1} \simeq \frac{\mathbb{C}[x]}{x-i} \times \frac{\mathbb{C}[x]}{x+i}$$

$$\simeq \mathbb{C} \times \mathbb{C}$$

Spec $\mathbb{C}$ is a variety connected, irred,

not geom. connected, irred

ex: k char p, $t \in k \setminus k^p \quad k[x]/_{x^p - t} = L$

$$L \otimes_k \bar{k} = \bar{k}[x]/x^p - t = \bar{k}[x]/(x-\alpha)^p = \bar{k}[\varepsilon]/\varepsilon^p$$

$\alpha^p = t$ in $\bar{k}$ $\quad \varepsilon = x - a$

Spec L reduced but not geom. reduced.

ex: $\mathbb{R}[x,y]/x^2+y^2 \rightsquigarrow \otimes_{\mathbb{R}} \mathbb{C} \quad \dfrac{\mathbb{C}[x,y]}{x^2+y^2} = \dfrac{\mathbb{C}[x,y]}{(x+iy)(x-iy)}$

domain

reducible.

irreducible, not geom. irreducible.

"Components of Varieties"

if $X = X_1 \sqcup \dots \sqcup X_n$ see this via global idempotents

$X = (X, \mathcal{O}_X)$

a constant fcn on a compnt $\rightsquigarrow$ global fcns $/k = \bar{k}$

$(\lambda, 0, \dots, 0)$

$\rightsquigarrow$ global idempotents $e_i \in \Gamma(\mathcal{O}_X)$

" e_i + galois action = notion of components"

Def E/k (k-alg) is étale if $E \simeq \prod E_i$

where E_i/k is a sep finite field ext.

Exercise: alt defn: E/k étale $\iff$ $E \otimes_k \bar{k} \simeq \prod^n \bar{k}$ $\circlearrowleft$ Gal k

$$\mathbb{C} \simeq \frac{\mathbb{R}[x]}{x^2+1} \longrightarrow \mathbb{C}$$

$x \mapsto i$, $x \mapsto -i$ — field pt eq.

$\frac{\mathbb{C}[x]}{x^2+1}$ $x \mapsto i$, $x \mapsto -i$ — not

Remark: If $E_1, E_2 \subseteq A$ k-algs w/ E_i/k étale then the subalg gen by E_1 & E_2 is étale.

$(E_1 \otimes_k E_2)_{\bar{k}} \rightsquigarrow$ étale

quot. of étale $\rightarrow$ étale

subalg. gen is a quot. of $E_1 \otimes_k E_2$

Exercise: actually prove this.

$Et(A) = \langle E \mid E \subset A \text{ étale} \rangle_{(\text{alg. by})}$

morally: if this keeps growing, only many independent idempotents contradict Noeth for $A_{\bar{k}}$

exercise: show $Et(A)$ is étale if A/k f.g.

$\Rightarrow$ if X/k finite type $\exists\, E \subset \Gamma(\mathcal{O}_X)$ max'l étale

<u>Def</u> $\pi_0 X \equiv \operatorname{Spec} E$

$X \to \operatorname{Spec} E$

$X_L \equiv X \times_{\operatorname{Spec} k} \operatorname{Spec} L$

<u>Proposition</u> if X a f.t.type k-scheme, L/k field ext

then $\pi_0(X_L) = (\pi_0 X)_L$

$(\pi_0 X) \times_k (\pi_0 Y) \simeq \pi_0(X \times_k Y)$

$\boxed{\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \text{ not connected}}$

$\Rightarrow \bar{k}[(\pi_0 X)_{\bar{k}}] = \prod^n \bar{k}$ $\qquad n = \dim_{\bar{k}} \bar{k}(\pi_0 X)_{\bar{k}} = \dim_k k[\pi_0 X]$

$\quad \|$

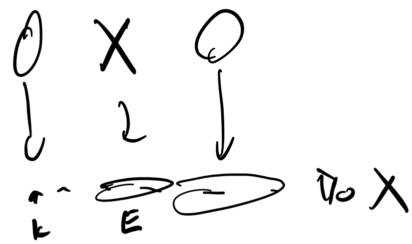
$\pi_0 X_{\bar{k}}$ = biggest étale in $\Gamma(\mathcal{O}_{X_{\bar{k}}})$ $= \bar{k}^{\max \# \text{ of idempotents in } \Gamma(\mathcal{O}_{X_{\bar{k}}})}$

étale$/\bar{k} = \bar{k} \times \dots \times \bar{k}$ $\quad = \bar{k}^{\# \text{components of } X_{\bar{k}}}$

<u>Cor</u>: if $\lambda \in \pi_0 X(k)$ then set $X_\lambda = \pi^{-1}\lambda$

via $\pi : X \to \pi_0 X$

then X_λ is geom. <u>connected.</u>



Cor: if G/k is a f.type gp scheme G_0 = connected comp. cont'g e

then G_0 is geom. connected

an
$$\begin{tikzcd} G_0 \times G_0 \arrow[r, dashed] \arrow[d, phantom, "\cap"] \arrow[dr] & G_0 \arrow[d, phantom, "\subset"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

$e \in G_0(k)$ $\qquad \pi : G \to \pi_0 G$

$\qquad\qquad e \mapsto$ a k-pt.