Alg. gps $\qquad$ Varieties$/_k$

$G \times G \to G$ gp structre.

$$\begin{tikzcd}
\text{Var.}/\mathbb{C} & & \text{Sch}/_{R} \arrow[ll, bend right] \\
k=\mathbb{C} \quad k=\mathbb{Q} & & \\
k=\mathbb{C}(x) & & \text{Sch}/_{S} \arrow[ll, squiggly]
\end{tikzcd}$$

Last time:

$$E = k[x]/_{x^2-a}$$

$$E \otimes_k \bar{k} = \bar{k}[x]/_{x^2-a} = \begin{cases} \bar{k} \times \bar{k} & \text{if } \operatorname{char} \neq 2 \\ \bar{k}[\varepsilon]/_{\varepsilon^2} & \text{if } \operatorname{char} = 2 \\ \quad \varepsilon = x - \alpha & \end{cases}$$

$$\alpha^2 = a \in \bar{k}$$

$$E = k[x]/_{x^6-a} = \begin{cases} \bar{k}^6 & \text{if } \operatorname{char} k \nmid 6 \\ \left(\bar{k}[\varepsilon]/_{\varepsilon^2}\right)^3 & \text{if } \operatorname{char} k = 2 \\ \left(\bar{k}[\varepsilon]/_{\varepsilon^3}\right)^2 & \text{- - -} \quad 3 \end{cases}$$

$\exists E/k$ ind ext $\quad E \otimes_k \bar{k} \simeq \bar{k}[\varepsilon]/\varepsilon^2 \times \bar{k}$

Lem E/k fin'd ext is separable $\iff E \otimes_k \bar{k} \simeq \bar{k}^n$

$n = [E:k]$

Pf: $E = k[x]/f$

sep $\iff$ f has dist roots in some sp field

$\Updownarrow$

— — — alg closure

$$\bar{k}[x]/f = \frac{\bar{k}[x]}{\prod(x-\alpha_i)} \underset{\text{CRT}}{\simeq} \bar{k}^n$$

conversely if $E \otimes_k \bar{k} \simeq \bar{k}^n \xrightarrow{\pi_i} \bar{k}$ have n distinct field pts of Spec $E_{\bar{k}}$

consider
$$\begin{tikzcd} E \arrow[r] \arrow[rr, bend right] \arrow[d, equal] & \bar{k}^n \arrow[r] & \bar{k} \\ k[x]/f & x \arrow[r, mapsto, "p_i"] & \alpha_i \end{tikzcd}$$

α_i roots of f in $\bar{k}$

$\Rightarrow \alpha_i$'s distinct.

$\Rightarrow$ separable.

alt. can show if not sep $\Rightarrow$ rep root then $\frac{\bar{k}(x)}{f}$ have a nilp

$$\frac{\bar{k}[x]}{\prod (x-\alpha_i)^{n_i}} \simeq \frac{\bar{k}[x]}{(x-\alpha_i)^{n_i}} \times \cdots$$

$\nearrow$

$\varepsilon = X - \alpha_i$ ε nilp.

$\simeq \bar{k}$

Reminder: $E/_k$ étale if $E \otimes_k \bar{k} \simeq \bar{k}^n$

or equiv $E \simeq \prod E_i$ $E_i/_k$ sep.

Rem: if $E_1, E_2/_k$ étale $\Rightarrow E_1 \otimes_k E_2$ étale

$$(E_1 \otimes_k E_2) \otimes_k \bar{k} \simeq (E_1 \otimes_k \bar{k}) \otimes_{\bar{k}} (E_2 \otimes_k \bar{k})$$

$$\bar{k}^n \otimes_{\bar{k}} \bar{k}^m \simeq \bar{k}^{nm}$$

$\rightsquigarrow$ if A Noeth, $\exists$ maxl étale $\pi_0 A \subseteq A$

if X scheme, $\pi_0 X \equiv \operatorname{Spec}(\pi_0 \Gamma(X, \mathcal{O}_X))$

$$\pi_0 A \longrightarrow A \qquad X \longrightarrow \pi_0 X$$

$$\begin{tikzcd} \overset{\pi_0 A}{E} \arrow[r, phantom, "\subset"] \arrow[d] & A \arrow[d] \\ \bar{E} \arrow[r, phantom, "\subset"] & B \end{tikzcd}$$

$\overline{\pi_0 A} \subset \pi_0 B$

Prop: $\pi_0(X \times Y) = \pi_0 X \times \pi_0 Y$

$(\pi_0 X)_L = \pi_0 X_L$

Observe: $\pi_0(X)(k) \neq \emptyset$

$\Downarrow$

$\pi_0 X = \prod E_i$ $\quad E_i/k$ sep

$\searrow k \qquad E_i \to k$

$k[\pi_0 X] = \textcircled{k} \times \prod E_i = \operatorname{Spec} k \times \square$

$\pi_0 X \times \pi_0 X$

$(\operatorname{Spec} k \times \operatorname{Spec} k) \cup (\operatorname{Spec} k \times \square) \times \dots$

$\operatorname{Spec} k$

Def X is geom. connected if $\pi_0 X \cong \operatorname{Spec} k$

$$\begin{tikzcd}
G \times G \arrow[r] \arrow[d] & G \arrow[d] \\
\pi_0(G \times G) \arrow[r] \arrow[d, equal] & \pi_0 G \\
\pi_0 G \times \pi_0 G &
\end{tikzcd}$$

$\pi_0 G$ also a gp sheaf

$G \to \pi_0 G$

$e \in G(k) \to \pi_0 G(k)$

let G_0 = connected comp of e.

this is geom connected.

connected

$$\begin{tikzcd}
G_0 \times G_0 \arrow[r] & G \\
e, e \arrow[r, mapsto] \arrow[u, phantom, "\in" rotate=90] & e
\end{tikzcd}$$

image of connected is connected

$\Rightarrow m(G_0 \times G_0)$ is in the connected comp. of G contg e.

$= G_0$

$\Rightarrow G_0 \subset G$ is a sub-gp.-scheme

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1 dim'l gps / $k=\bar{k}$

Suppose G is a smooth affine variety / $k=\bar{k}$ w/ gp struc of dim 1.

$G \subseteq X$ X sm. proj curve

and G acts on itself, so also acts on X

if $g \in G(k)$ $m_g : G \to G \subset X$

$$\begin{tikzcd}
 & pt \arrow[r, mapsto] & g & \\
G \arrow[r, phantom, "\cong"] & pt \times G \arrow[r] & G \times G \arrow[r, "m"] & G \\
h \arrow[rrr, bend right] & (*,h) \arrow[r, hook, mapsto] & (g,h) \arrow[r, mapsto] & gh
\end{tikzcd}$$

$m_g : X \dashrightarrow X$ valuative criterion for properness

extends to a morphism

$$\begin{tikzcd}
X \arrow[r, "g"] & X \arrow[r, "g^{-1}"] & X \\
G \arrow[u, phantom, sloped, "\subset"] \arrow[r] \arrow[rr, bend right, "id"'] & G \arrow[u, phantom, sloped, "\subset"] \arrow[r] & G \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

$G(k) \to Aut(X)$ dense orbit

$\hookrightarrow$

$g, g' \in G(k)$

$e \in G(k) \subseteq X(k)$

$g \cdot e = g' \cdot e$

$\Rightarrow \mathrm{Aut}(X)$ are infinite, $\Rightarrow$ genus $X = 0$ or 1.

$\mathbb{P}^1$ elliptic.

if affine, genus 1 $\quad G = \text{Elliptic} \setminus \text{finite set of pts.}$

$\Rightarrow$ finite aut gp for G.

K finite $\Leftrightarrow$ auts elliptic fixing 1 pt finite

$$\begin{tikzcd}
K \arrow[r] & G(k) \arrow[r] & \text{perms of missing pts.}
\end{tikzcd}$$

$\uparrow$ finite index $\qquad\qquad$ $\uparrow$ finite

genus 1 [illegible]

$G \subseteq \mathbb{P}^1 \qquad G(k) \hookrightarrow \mathrm{Aut}\,\mathbb{P}^1$ "linear fractionals"

$$z \mapsto \frac{az+b}{cz+d}$$

$G = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1$?

$= \mathbb{P}^1 \setminus \{0,1\}$

$= \mathbb{P}^1 \setminus \ldots$

$$\begin{tikzcd}
G(k) \arrow[r, hook] \arrow[d, equal] & \mathrm{Aut}(\mathbb{P}^1) \\
\mathbb{P}^1 \setminus S & \{\sigma \in \mathrm{Aut}\,\mathbb{P}^1 \mid \sigma S = S\}
\end{tikzcd}$$

$G(k) \hookrightarrow \{\sigma \in \mathrm{Aut}\,\mathbb{P}^1 \mid \sigma S = S\}$

$$1 \to \mathrm{Stab}\, S \to G(k) \to \mathrm{Perm}\, S \to 1$$

$\# S \geq 3 \quad \mathrm{Stab}\, S = \{1\}$