No class next week

Plan:

- Finish classification of 1dim'l gps
- define rep's, will show affine alg gp's / fields admit faithful lin reps

  if smooth $\Rightarrow$ closed subgp of GL_n

  "Linear alg gp"

  fill in some facts about (connectedness) & smoothness

  morphisms between gps
- head towards structure thy, starting w/ tori (étale...)

Last time: G 1-dim'l gp variety / $k = \bar{k}$

$X = \bar{G}$ sm. proj. curve

action $G \circlearrowright G$ extends to a faithful action $G \circlearrowright X$

G permutes points of $X \setminus G$

$$1 \to \underset{\text{finite}}{\mathrm{Stab}} \longrightarrow \underset{\text{infinite}}{G(k)} \longrightarrow \underset{\text{finite}}{\mathrm{Perm}(X \setminus G)}$$

$X = \mathbb{P}^1$ $\quad$ $\mathrm{Stab}(p_1, \dots, p_n)$ in $\mathbb{P}^1$ is going to be (e)

if $n \geq 3$

$\Rightarrow G = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1$

or $\mathbb{P} \setminus \{0, \infty\} = \mathbb{A}^1 \setminus \{0\}$

$G(k) \hookrightarrow \mathrm{Aut}(\mathbb{P}^1) = PGL_2(k)$ $\quad$ $z \mapsto \dfrac{az+b}{cz+d}$

which have fixed translates fix ∞?

$z \longmapsto az + b$

if $a \neq 1$ note $\quad$ $z = az + b$

$(1-a)z = b$

$z = \dfrac{b}{1-a}$ fixed pt.

$\Rightarrow G(k) \subseteq \{ z + b \} \subseteq \left(\begin{matrix} a & b \\ c & d \end{matrix} \,\middle|\, \begin{matrix} c = 0 \\ a = 1 \\ d = 1 \end{matrix} \right)$

$\simeq \mathbb{A}^1$

$G \xrightarrow{\sim} {}''\mathbb{A}^1{}'' = \mathbb{G}_a$

- $\mathrm{Aut}(\mathbb{P}^1, \text{fix } \infty, \text{no } \cancel{\text{fixed}} \text{ pts}) \simeq \mathbb{A}^1$ $\mathbb{G}_a$

functor is rep by

$G \to \mathrm{Aut}(\mathbb{P}^1, \dots)$ bijective on pts $\Rightarrow$ iso of varieties.

$(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1 \longrightarrow \mathrm{Aut}(\mathbb{P}^1, \text{fix } \infty)$

$\mathbb{P}^1(R) = \{ L \subseteq R \oplus R \mid L \text{ rk } 1 \text{ proj. module s.t. } \exists Q \text{ w/ } Q \oplus L \cong R^2 \}$

$$\begin{tikzcd} L \arrow[r, hook, "\psi"] \arrow[d, "\varphi", "\wr"'] & R^2 \\ L \arrow[r, hook, "\psi"'] & R^2 \arrow[d, "\varphi"] \\ & R^2 \end{tikzcd}$$

$(x,y) \mapsto (ax+by, y)$

$a, b \in (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$

$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \varphi$

Or: fix 2 pts $0, \infty$

$z \mapsto \dfrac{az+b}{cz+d}$

fix ptwise: $z \mapsto az$

~~swap: $z \mapsto \frac{a}{z}$~~

$z = \frac{a}{z} \rightsquigarrow z^2 = a \quad z = \pm\sqrt{a}$

$\Rightarrow \quad G \simeq \{ a \in \mathbb{A}^1 \setminus \{0\} \} \simeq \mathbb{G}_m$

G connected group-variety/k

$$1 \to H \to G \to A \to 1$$ "Chevalley"

$\exists$ s.e.s. where H is affine (in fact linear) & A projective. "Abelian variety"

if G smooth connected k-gp finite type "anti-Chevalley"

$$1 \to Z \to G \to G^{aff} \to 1$$

Z: commutative. connected

G^{aff}: affine (smooth) LAG

$$1 \to T \to Z \to A \to 1$$

T: linear "tori"

A: abelian

1

$\uparrow$

$\mathcal{G}$ = gp. f.type sep. reduced scheme

$$1 \to G_0 \to G \to \pi_0 G \to 1$$

semisimple $\qquad$ finite $\qquad$ $k = \bar{k}$

"G_a stuff" "unipotent rad"

$$\begin{tikzcd}
 & & 1 & 1 \arrow[d] & & \\
 & & G^{ad} \arrow[u] & G^{ss} \arrow[l] \arrow[d] & Z \arrow[l] & 1 \arrow[l] \\
1 \arrow[r] & R_u G_0 \arrow[r] & G_0 \arrow[r] \arrow[u] & G_0^{red} \arrow[r] \arrow[d] & 1 & \\
 & & RG_0 \arrow[u] & \text{Tori} \arrow[d] & & \\
 & & 1 \arrow[u] & 1 & &
\end{tikzcd}$$

"max'l solvable"

building blocks	unipotent $R_u G_0$	G_a
	tori	G_m
	semisimple	SL_2

Representations

Def A functorial rep. of G is a nat. trans. of gp functors

$$G \to GL_n$$

Def A functorial G-module is, for each R, an R-module $M(R)$ w/ an action $G(R) \curvearrowright M(R)$ as R-module auts together w/ identifications for $R \to R'$

$$M(R) \otimes_R R' \xrightarrow{\sim} M(R') \quad \leftarrow \text{making } M \text{ into a functor}$$

& such that

$$\begin{tikzcd}
G(R) \times M(R) \arrow[r] \arrow[d] & M(R) \arrow[d] \\
G(R') \times M(R') \arrow[r] & M(R')
\end{tikzcd}$$

commutes.

Lemma: functorial G-mod & basis for $M(k)$ = functional G-rep.

Def A $k[G]$-comodule N is a linear map

$N \to k[G] \otimes_k N$ s.t.

$$\begin{tikzcd}
k[G] \otimes N \arrow[r, "\mathrm{id} \otimes c"] & k[G] \otimes k[G] \otimes N \\
N \arrow[u, "c"] \arrow[r, "c"'] \arrow[ur, phantom, "\text{commutes}"] & k[G] \otimes N \arrow[u, "\Delta \otimes \mathrm{id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
G \times G \arrow[d, "m"] & k[G] \otimes k[G] \\
G & k[G] \arrow[u, "\Delta"'] \arrow[d, "\varepsilon"] \\
\operatorname{Spec} k \arrow[u, "e"'] & k
\end{tikzcd}$$

$$\begin{tikzcd}
k[G] \otimes N \arrow[r, "\varepsilon \otimes \mathrm{id}"] & N \\
N \arrow[u, "c"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
G \times N \arrow[d, "m"] & N \arrow[l, "{e, n}"'] \arrow[dl] \\
N &
\end{tikzcd}$$

__Punchline:__ comodule $N \longleftrightarrow$ funct. G-module M

$$N = M^{sf}$$