Back to end of classification:

$$\begin{tikzcd} G(k) \arrow[r, hook] & \mathrm{Aut}\,\mathbb{P}^1 \end{tikzcd}$$

$$\mathbb{G}_a \subset \mathrm{Aut}\,\mathbb{P}^1 \supset \mathbb{G}_m$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \mathbb{G}_a \\ & \mathbb{G}_m \end{tikzcd}$$

gp hom / morphisms of schemes injective on $\bar{k} = k$ points.

$$\begin{tikzcd} U \arrow[r] \arrow[d, phantom, "\cap"] & V \arrow[d, phantom, "\cap"] \\ \mathbb{P}^1 \arrow[r, dashed] \arrow[r, Rightarrow] & \mathbb{P}^1 \\ y & x \\ k[x][g^{-1}] & k[x][f^{-1}] \arrow[l] \\ ? & x \arrow[l, mapsto] \end{tikzcd}$$

map between two opens in $\mathbb{P}^1$

inj. $\Rightarrow$ iso.

proper & inj $\Rightarrow$ surj.

f.fields field ext. $\quad k(y) \mid k(x)$ $\quad$ assume $k(y) = k(x)[T]/f(T)$

if f has $\deg \geq 2$

write down an example look same at gen pt.

$$\frac{k[x][g^{-1}][T]}{f(T)} \Big| \; k[x][g^{-1}] \qquad \begin{matrix} V' \\ \downarrow \\ U' \end{matrix} \qquad \begin{matrix} \mathbb{P}^1 \\ \vdots \\ \mathbb{P}^1 \end{matrix} \qquad \begin{matrix} V \\ \vdots \\ U \end{matrix}$$

$X \overset{f}{\underset{g}{\rightrightarrows}} Y$ Y sep   agree on closed sub of X

$$\begin{tikzcd}
\{f=g\} \text{ locus} \arrow[r] \arrow[d] & X \arrow[d, "f \times g"] \\
Y \arrow[r, "\Delta"', "\text{closed}"'] & Y \times Y
\end{tikzcd}$$

Representations & comodules

Def a $k[G]$-comodule N is a k-v.s. N

w/ $N \xrightarrow{c} k[G] \otimes_k N$ s.t.

$$\begin{tikzcd}
N \arrow[r, "c"] \arrow[d, "c"] & k[G] \otimes N \arrow[d, "\Delta \otimes id"] \\
k[G] \otimes N \arrow[r, "id \otimes c"'] & k[G] \otimes k[G] \otimes N
\end{tikzcd}$$

commutes.

& counit acts as a "coidentity"

Given a R-mod N, form $N^* = \mathrm{Hom}_R(N, R)$

Def $S^\bullet(N)$ = free alg. gen. by N as an R-submod.

$R \oplus N \oplus (S^2 N) \oplus (S^3 N)$

$S^n N = \overbrace{N \otimes \cdots \otimes N}^{n} /$ identify permutations

$x_1 \otimes \cdots \otimes x_n \sim x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$

$\sigma \in S_n$

given M/A

$\rightsquigarrow$ can construct a sheaf of mods / Spec A $\tilde{M}$

this can be thought of as a functor from

basic (affine) opens in A

$$\operatorname{Spec} A_f \longmapsto \tilde{M}(\operatorname{Spec} A_f) = M \otimes_A A_f \text{ an } A_f\text{-module}$$

$$\begin{matrix} A_f \\ | \\ A \end{matrix} \text{ ring ext} \rightsquigarrow \tilde{M}(\operatorname{Spec} A_f) \text{ an } A_f \text{ module.}$$

given M/k get a funct from (works if M is "reflexive")

$\begin{matrix} R \\ | \\ k \end{matrix}$ k-alg $\rightsquigarrow$ R-mod$_{\text{r sets}}$ $M^{**} \cong M$

$R \longmapsto M(R)$

want a "scheme analog for M"

$$M(\;) = \operatorname{Spec} S^\bullet(M^*)$$

$$M(R) = \operatorname{Hom}_{k\text{-alg}}(S^\bullet(M^*), R)$$

$$= \operatorname{Hom}_{k\text{-mod}}(M^*, R)$$

if M is finitely presented

$$= \operatorname{Hom}_{R\text{-mod}}(M^* \otimes_k R, R)$$

$$= \operatorname{Hom}_{R\text{-mod}}((M \otimes_k R)^*, R)$$

$$= (M \otimes_k R)^{**} = M^{**} \otimes_k R = M \otimes_k R$$

<u>Def</u> A functorial G-module was

a functor $M : R \longrightarrow M(R)$

$M(R) \cong M(k) \otimes_k R$ w/ natural G action

(i.e. a nat. trans. functors $G \times M \longrightarrow M$)

$$G(R) \times M(R) \longrightarrow M(R)$$

s.t. each gives $G(R)$ acting as auts of $M(R)$

Def A functorial G-rep is a nat trans (of gp functors)

$$G \to GL_n.$$

Last time: G rep = G mod + basis for $M(k)$

This time = (if $M(k)$ f.d.) f.d. $k[G]$-comodule

$$N = M(k)^*$$

given M f.d. v.space/k & $k[G]$ comodule structure on

$$M^* = N$$

$\rightsquigarrow$ get actions $G(R) \circlearrowright M(R)$ (funct. G-mod)

$$\begin{tikzcd} M^* \arrow[r] \arrow[d, "\exists!"'] & k[G] \otimes M^* \subset k[G] \otimes_k S^\bullet(M^*) \\ S^\bullet(M^*) \arrow[ur] & \end{tikzcd}$$

univ prop

Recall: $\mathbb{A}(M) \equiv \operatorname{Spec} S^\bullet(M^*)$

$$\mathbb{A}(M)(R) = \cdots = M \otimes_k R$$

$$\operatorname{Spec}(k[G] \otimes S^\bullet M^*) \longrightarrow \operatorname{Spec} S^\bullet M^*$$

$$G \times \mathbb{A}(M) \longrightarrow \mathbb{A}(M)$$

evaluate at R.

$$G(R) \times M(R) \longrightarrow M(R)$$

<u>Goal</u>: show that G affine gp scheme/k

admits a faithful f.dim'l representation

f.d $k[G]$-comod or

$G \to GL_n$

idea:

$G \circlearrowright k[G]$ find $V < k[G]$ f.d. vspace w/ G v.c.

observe $k[G]$ is a $k[G]$-comodule.

$V \le k[G]$ subcomodule, f.d'l $\rightsquigarrow$ f.d. rep of G.

"$k[G]$-subcomodule."

<u>Goal</u>: find a "big enough" G-stable subspace

- <u>Remark</u>: if $V, W \le k[G]$ are subcomodules

$\Rightarrow$ $V + W$ is also a subcomod

- also $k[G]$ = a directed limit union of increasing subspaces of f.d subspaces

<u>will show</u>: if $x \in k[G]$ $\exists$ $V \ni x$ f.d. subcomod.

$$\overbrace{k[G]} \xrightarrow{\text{Hopf alg.}} k[G] \otimes \overbrace{k[G]}^{\text{comod.}} \longrightarrow k[G]$$

$$x \longmapsto \sum v_i \otimes x_i$$

let $V = \langle x_i \rangle$

$x \in \langle x_i \rangle$ count