Last time

Goal: show if G/k is an affine gp scheme then G admits a faithful representation

$G \hookrightarrow GL_n$ (eventually: as a closed subgp)

Strategy:

- Observe that $k[G]$ is a comodule for $k[G]$
- If $x \in k[G]$ then $\exists \overset{x \in}{V} \subseteq k[G]$ finite dim'l s.t. V is a sub-$k[G]$ comodule
- If $V, W \subseteq k[G]$ are sub comodules then so is $V+W$.
- Observe if $W \subseteq k[G]$ f.dim'l $\exists W \subseteq V$ f.dim'l subcomodule

(choose $x_i \in W$ basis, V_i subcomodule f.d containing x_i, $W \subseteq \sum V_i$)

- Can write (since $k[G]$ is a f.g. algebra)

$$k[G] = \bigcup_{i=0}^{\infty} W_i \qquad W_i \subseteq W_{i+1}$$

$W_0 = \langle x_1, \ldots, x_n \rangle$ x_i's generating set

$$W_i = \sum_{j \leq i} W_0^j$$

$\Rightarrow$ write $k[G] = \bigcup_{i=0}^{\infty} V_i$ $\quad V_i$ subcomod

for each i $\exists\, \tilde{V}_i \supseteq W_i$ $\quad \tilde{V}_i$ subcomod.

$V_i = \sum_{j \le i} V_j$

Main sticky points

if $x \in k[G]^{comod}$ want $x \in V \subseteq k[G]$ f.d. subcomod.

$$k[G]^{comod} \xrightarrow{\Delta} k[G] \otimes k[G]^{comod}$$

$$x \longmapsto \sum_{i=1}^{n} v_i \otimes x_i \qquad \text{claim } V = \langle x_i \rangle \text{ is a subcomod.}$$

$x \in V$? $\qquad \varepsilon : k[G] \to k \qquad e : \operatorname{Spec} k \to G$

$$\begin{tikzcd}
k[G] \arrow[r, equal] & k \otimes k[G] \\
 & k[G] \otimes k[G] \arrow[u, "\varepsilon \otimes id"'] \\
 & k[G] \arrow[u, "\Delta"'] \arrow[uul, bend left, "id"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum \varepsilon(v_i) \otimes x_i \\
\sum v_i \otimes x_i \arrow[u] \\
x \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
G \cong G \times \operatorname{Spec} k \arrow[d, "id \times e"] \arrow[dd, bend right, "id_G"'] \\
G \times G \arrow[d, "m"] \\
G
\end{tikzcd}$$

$x = \sum \varepsilon(v_i) x_i$

(to be continued)

Connectedness & smoothness

regular vs smooth

Def A local ring (Noeth.) is regular if $\dim A = \dim_{A/m} m/m^2$

$$\begin{array}{cccc} f\in A \rightsquigarrow & f-f(0)\in m & m^2 & m/m^2 \\ \downarrow & \downarrow & \downarrow \text{derivative} & \downarrow \text{derivative} \\ f' & f' & 0 & ? \end{array}$$

regularity $\Leftrightarrow$ tangent space (dual to m/m^2) has the expected size.

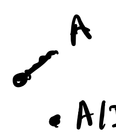
$Z(xy)$ $\quad k[x,y]/xy$ $\quad (x,y)$

$(x,y)/(x^2,xy,y^2)$

x, y

Def $X \xrightarrow{f} Y$ formally smooth (map of schemes / k-schemes) (k functs)

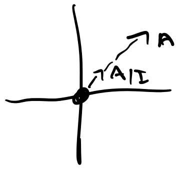
if given A k-alg w/ ideal $I \triangleleft A$ s.t. $I^2=0$



then any diagram

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

$k[\varepsilon]/\varepsilon^2$



for example:

$$\begin{tikzcd} \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[r] \arrow[d] & \operatorname{Spec} k[x,y]/xy \arrow[d] \\ \operatorname{Spec} k[\varepsilon]/\varepsilon^3 \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$$\varepsilon, \varepsilon \longleftarrow\!\shortmid\; x, y$$

$$\begin{tikzcd} k[\varepsilon]/\varepsilon^2 & k[x,y]/xy \arrow[l] \arrow[dl, dashed, "\times"] \\ k[\varepsilon]/\varepsilon^3 \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

$$\varepsilon + \alpha\varepsilon^2, \ \varepsilon + \beta\varepsilon^2$$

$k = \mathbb{F}_p(t)$

$L = k(\sqrt[p]{t})$

Spec L is regular (0 dim'l
$m = 0$
$m/m^2 = 0$
0 generates ✓)

But Spec $L \to$ Spec k not formally smooth

$\alpha \in \bar{k}$ $\quad (\alpha+\varepsilon)^p = t$

$\alpha + \varepsilon \longleftarrow\!\shortmid\; \alpha$

$k(\alpha) \quad \alpha^p = t$

$$\begin{tikzcd} \bar{k}[\varepsilon]/\varepsilon^p & L \arrow[l] \arrow[dl, dashed, "\times"] \\ \bar{k}[\varepsilon]/\varepsilon^{p+1} \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

α

$\alpha + \varepsilon + (\varepsilon^{?}\gamma) \longleftarrow$

$$\begin{tikzcd} \operatorname{Spec} \bar{k}[\varepsilon]/\varepsilon^m \arrow[r] \arrow[d] & \operatorname{Spec} L \arrow[d] \\ \operatorname{Spec} \bar{k}[\varepsilon]/\varepsilon^n \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$\alpha^p + \varepsilon^p \notin t$

<u>Def</u> $X \xrightarrow{f} Y$ is smooth if f is finite presentation & formally smooth

mostly interested in $Y = \operatorname{Spec} k$

for $X/_k$ k-scheme, will often consider infinitesimal structure near a point $x \in X(k)$

<u>Recall</u>: $\mathcal{O}_{X,x}$ local ring at x, $m_{X,x}$ max'l ideal

$\hat{\mathcal{O}}_{X,x} = m_{X,x}$-adic completion of $\mathcal{O}_{X,x}$ complete local ring.

<u>Thm</u>: If k a field, A a complete local ring w/ max'l m and $A/m \cong k$ then TFAE

- A regular (i.e. $\dim A = \dim_k m/m^2$)
- $A = k[[x_1, \ldots, x_n]]$ $n = \dim A$
- given R w/ ideal J, $J^2 = 0$ then any $A \to R/J$ k-alg map lifts to $A \to R$ k-alg map.

why is SL_n (& GL_n) smooth$/_k$?

if $J \trianglelefteq R$ $J^2=0$, $SL_n(R) \twoheadrightarrow SL_n(R/J)$

if $T \in GL_n(R/J) \subseteq M_n(R/J) = (R/J)^{n^2}$

lift to $\tilde{T} \in M_n(R)$

$$\det(\tilde{T}) \in R \longrightarrow R/J$$
$$\longmapsto \det(T) \in (R/J)^*$$

but J nilpotent so $\det(\tilde{T})$ is invertible.

in SL_n case

$\det(\tilde{T}) \in 1 + J$ $\begin{pmatrix} 1+x & 1+y & \cdots \end{pmatrix}$

observe: if $S = 1 + de_{11}$

$$\det(1+S) = 1+d$$

$$\tilde{\tilde{T}} = \tilde{T}(1+S) \xrightarrow{\det} (1+d)(1-d) = 1-d^2 = 1$$

$d = -\det$