Math 3120, Linear Algebra, Spring 2026, Exam 1 practice sheet

1. Consider the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$.

   (a) Exhibit a sequence of row operations which puts A into reduced row echelon form.

   (b) Show how to write A as a product of elementary matrices (matrices which correspond to elementary row operations).

2. Consider the following row operations on a system of 2 equations:

   - exchange the two equations,
   - multiply the first equation by 1/3,
   - subtract the first equation from the second,
   - multiply the second equation by 2,
   - add the second equation to the first.

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \mapsto \begin{pmatrix} R_1 \\ R_2 - R_1 \end{pmatrix}$$

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \mapsto \begin{pmatrix} R_1 - R_2 \\ R_2 \end{pmatrix}$$

   (a) Suppose that A is a 2×2 matrix which, after applying the above row operations in order, yields the identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What is A^{-1}?

   (b) If A and B are both 2×2 matrices such that the application of these row operations puts them both into reduced row echelon form, does it follow that $A = B$? Why or why not?

if $A, B \neq 0$

"trick question" w/ valid "clever & [illegible]" & "very clever & long" solutions.

3. Consider the vectors $v_1 = (1,0,1), v_2 = (1,2,2)$ in $\mathbb{R}^3$. Describe all vectors which are perpendicular to both v_1 and v_2.

$x = (x_1, x_2, x_3)$

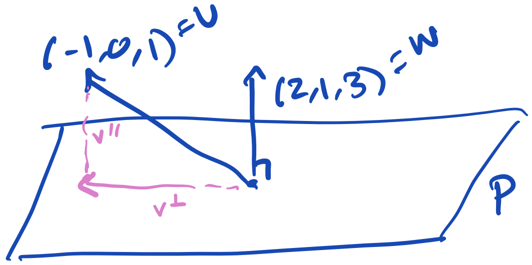
$x \cdot v_1 = 0 = x_1 + 0x_2 + x_3$

$x \cdot v_2 = 0 = x_1 + 2x_2 + 2x_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} x = 0$$

4. Let P be the plane in $\mathbb{R}^3$ consisting of those vectors which are perpendicular to the vector $(2,1,3)$.

(a) Find the vector in P which is closest to $v = (-1,0,1)$.



$v^\perp = v - v^{\parallel}$

$v = v^{\parallel} + v^\perp$

$v^{\parallel} = \lambda w$

$v^\perp \cdot w = 0$

difference between v & $v^\perp$

closest vector to v in P.

$v^{\parallel} = \frac{v \cdot w}{w \cdot w} w = \frac{(-1,0,1)\cdot(2,1,3)}{(2,1,3)\cdot(2,1,3)} w$

defines plane

$\left(\frac{-2+3}{4+1+9}\right)\cdot(2,1,3)$

$x^\perp = x - \underbrace{\frac{v \cdot x}{v \cdot v} v}_{x^{\parallel}}$

vector defining plane

$= x - x^{\parallel}$

(b) Calculate the distance between P and v.

$\| v^{\parallel} \|$

$w = (x,y,z) \in \mathbb{R}^3$ $D(w) = v \cdot w$

5. (a) Consider the vector $v = (2, 1, 3)$. What matrix represents the linear transformation $D : \mathbb{R}^3 \to \mathbb{R}^1$ sending a vector w to the dot product $v \cdot w$?

3x1 matrix

matrix for $D = [2 \;\; 1 \;\; 3]$ $D(x,y,z) = 2x + y + 3z$
$= (2,1,3)\cdot(x,y,z)$

(b) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by the matrix $\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$. If we denote the columns of T by w_1, w_2 show that

$= \text{range } T = \{Tx \mid x \in \mathbb{R}^2\}$

- Every vector u in the image of T can be written in the form $aw_1 + bw_2$.
- The vectors w_1 and w_2 are independent.

$T(x) = ?$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $Tx = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

independence: want to show: $\begin{bmatrix} 2a \\ a+b \\ a+3b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then a=0=b

$= \begin{bmatrix} 2x_1 \\ x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix} \stackrel{?}{=} aw_1 + bw_2$

(c) Give a matrix representation for the composition $D \circ T$.

$aw_1 + bw_2 = a\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2a \\ a+b \\ a+3b \end{bmatrix}$

$\{u \in \mathbb{R}^2 \mid D\circ T(u) = 0\}$

$\shortparallel$

nullspace

(d) The kernel of $D \circ T$ can be represented as the collection of vectors which are perpendicular to a vector w. What is such a vector w?

$D\circ T \longleftrightarrow [2 \;\; 1 \;\; 3]\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = [8 \;\; 10]$

$[8 \;\; 10] \cdot u = 0$

6. Consider the following sequence of linear transformations in $\mathbb{R}^2$

   - swap the x and y axes
   - rotate 90° counterclockwise
   - $(x, y) \mapsto (x, y - 2x)$
   - rotate 180°
   - swap the x and y axes

   Find a matrix representing the linear transformation obtained by doing this full sequence of operations in order.

7. Suppose we have an $m \times n$ matrix A whose columns correspond to vectors $v_1, \ldots, v_n$ and whose rows correspond to vectors $w_1, \ldots, w_m$. If all of the vectors v_i are parallel to each other, is the same necessarily true about the w_i's?

   Either explain why this must always be true, or provide a counterexample.

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \\ 3 & -3 \end{bmatrix} \qquad \begin{bmatrix} a & \lambda a \\ b & \lambda b \\ c & \lambda c \end{bmatrix}$$

rows all mults of $[a \;\; \lambda a]$

$\to b/a$

$\to c/a$

$$\begin{bmatrix} a_1 & \lambda_1 a_1 & \cdots & \lambda_m a_1 \\ a_2 & \lambda_1 a_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ a_n & \lambda_n a_n & & \lambda_m a_n \end{bmatrix}$$

$\to$ case 1

all a_i's $= 0$

$\Rightarrow$ matrix is 0

$\to$ case 2

some $a_i \neq 0$

get all other rows j

as $R_i \cdot a_j / a_i$

v = blueprint for columns

$$\left[\lambda_1 v \,\middle|\, \lambda_2 v \,\middle|\, \cdots \,\middle|\, \lambda_m v\right] = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} \lambda_1, \dots, \lambda_m \end{bmatrix}$$