

MATH 3120, Linear Algebra, Spring 2026, Exam 2

Review Sheet

1. Let A be an $n \times n$ matrix, and suppose $x_1, \dots, x_m$ is a basis for the space of solutions to $Ax = 0$. For a vector $v \in \mathbb{R}^n$ how would you find a vector v' which is a solution to $Ax = 0$, and as close as possible to v ?

matrix

2. Let A be a ~~linear transformation~~ whose columns are the vectors $w_1, \dots, w_n$ and whose rows are the vectors $v_1, \dots, v_m$. Let W be the span of the vectors $w_1, \dots, w_n$ and let U be the span of the vectors $v_1, \dots, v_m$. Let T is the linear transformation corresponding to A .

$$A = [w_1 | \dots | w_n] = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

$W = \langle w_1, \dots, w_n \rangle$
 $U = \langle v_1, \dots, v_m \rangle$

- (a) Show that the image of T is W

By definition the image of T is the vectors of the form $T(\vec{a})$ or $A\vec{a}$ for $\vec{a} \in \mathbb{R}^n$.
 and W is the set of vectors of the form $a_1 w_1 + \dots + a_n w_n$

So now you can just check $\text{image } T = \text{stuff of form } A\vec{a} = [w_1 | \dots | w_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 w_1 + \dots + a_n w_n$
 which is exactly what said W was. done. ✓

- (b) Show that the kernel of T is $U^\perp$, the set of vectors perpendicular to each vector in U .

By definition, the kernel of T is the vectors $\vec{a}$ such that $T(\vec{a}) = 0$.
 or in other words, the vectors $\vec{a}$ such that $A\vec{a} = 0$

Image of T is $T(\vec{a})$

use this representation
for part b)

$$T(\vec{a}) = Aa = A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad Aa = \begin{bmatrix} \overline{v_1} \\ \vdots \\ \overline{v_m} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \vec{a} \cdot v_1 \\ \vec{a} \cdot v_2 \\ \vdots \\ \vec{a} \cdot v_m \end{bmatrix}$$

$$Aa = \begin{bmatrix} | & & | \\ w_1 & \dots & w_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 w_1 + a_2 w_2 + \dots + a_n w_n$$

3. Suppose you are given vectors $v_1, \dots, v_m \in \mathbb{R}^n$ and a vector $a \in \mathbb{R}^n$.

Describe how you would find a matrix A and a vector b such that the solutions to $Ax = b$ exactly consists of those vectors $v \in \mathbb{R}^n$ such that the vector $v + a$ is perpendicular to each v_i .

4. Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 4 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Decide whether or not this matrix is invertible (hint: use the determinant!)

5. Let A be an $m \times n$ matrix, and suppose $x \in \mathbb{R}^n$ is a vector.

- (a) (stupid trick question) While it might not be the case that x is a solution to $Ax = 0$, can we always find a b such that x is a solution to $Ax = b$?

If $Ax=b$ is inconsistent
can we change b to make it
consistent?

- (b) If so, how can we find the smallest vector b so that $Ax = b$ has a solution?

as close as possible to b so that

6. Consider the vectors $v_1 = (3, 1, 0, 0)$, $v_2 = (1, 2, 1, 1)$, $v_3 = (2, 2, 0, 1)$. Let $V \subset \mathbb{R}^4$ be the span of these vectors. Check to see whether or not $(1, 0, 0, 0)$ is in V .

hint: use the determinant and the various computational shortcuts we have seen!

First solve for x in $BAx = -Ba$ then take the resulting x and plug into $Ax + a$ to answer.

7. Suppose T is a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^n$ given by a matrix A , and S is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^q$ given by a matrix B . Let $a \in \mathbb{R}^n$. How might you solve for a vector of the form $Ax + a$ which is as close as possible to a solution to the equation $By = 0$?

$$B(Ax + a) = 0$$

$$BAx = -Ba$$



if inconsistent, use least squares
 $(BA)^T(BA)x = -(BA)^T Ba$
 then soln will be an x such that BAx close to $-Ba$

8. Suppose T is a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^n$ given by a matrix A , and S is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^q$ given by a matrix B . Let $b \in \mathbb{R}^q$ be a vector. How might you solve for a vector of the form Ax which is as close as possible to a solution to the equation $By + b$?
- or $BAx + Ba$ is small as possible

9. Consider the vectors $(1, 2, 0)$ and $(1, 1, 1)$ in $\mathbb{R}^3$ and let $V \subseteq \mathbb{R}^3$ denote their span.

- find an orthonormal basis for V
- find a 3×3 matrix representing the projection onto V .

10. Suppose W and V are vector spaces, and $T : V \rightarrow W$ is a linear transformation. Suppose that T is injective.

def: $\langle, \rangle$ is an inner prod if

$$1. \langle v, w \rangle = \langle w, v \rangle$$

$$2. \langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$$

$$3. \langle \lambda v, w \rangle = \lambda \langle v, w \rangle \quad 4. \langle v, v \rangle \geq 0$$

only if $v = 0$

- (a) Show that if W has an inner product $\langle, \rangle$ then we may define an inner product on V by defining, for $v_1, v_2 \in V$, $\langle v_1, v_2 \rangle_V = \langle T(v_1), T(v_2) \rangle_W$.

$$\langle v, w \rangle_V = \langle w, v \rangle_V?$$

Def: $\langle v, w \rangle_V = \langle T(v), T(w) \rangle_W = \langle T(w), T(v) \rangle_W = \langle w, v \rangle_V$

$\langle w, \cdot \rangle$ is an inner prod

- (b) Show that if T is not injective, this will not give an inner product.

$$\langle v, v \rangle_V \geq 0?$$

$$\langle T(v), T(v) \rangle_W \geq 0$$

(a) To show that we have an inner product on V , we need to check all 4 axioms hold.

Checking axiom 1: is $\langle v, w \rangle_V \stackrel{?}{=} \langle w, v \rangle_V$

$$\text{Def: } \langle v, w \rangle_V = \langle T(v), T(w) \rangle_W$$

since $\langle, \rangle_W$ is an inner product,

$$\langle T(v), T(w) \rangle_W = \langle T(w), T(v) \rangle_W$$

$$\begin{aligned} \text{So } \langle v, w \rangle_V &= \langle T(v), T(w) \rangle_W = \langle T(w), T(v) \rangle_W \\ &\stackrel{\text{by def}}{=} \langle w, v \rangle_V. \end{aligned}$$

Checking axiom 2: is $\langle v_1 + v_2, w \rangle_V \stackrel{?}{=} \langle v_1, w \rangle_V + \langle v_2, w \rangle_V$

$$\text{we have } \langle v_1 + v_2, w \rangle_V = \langle T(v_1 + v_2), T(w) \rangle_W$$

$$(\text{since } T \text{ is a linear transformation}) = \langle T(v_1) + T(v_2), T(w) \rangle_W$$

$$\begin{aligned} (\text{since } \langle, \rangle_W \text{ satisfies axiom 2}) &= \langle T(v_1), T(w) \rangle_W + \langle T(v_2), T(w) \rangle_W \\ &= \langle v_1, w \rangle_V + \langle v_2, w \rangle_V \end{aligned}$$

Checking axiom 3: is $\langle \lambda v, w \rangle_V \stackrel{?}{=} \lambda \langle v, w \rangle_V$

$$\text{well, } \langle \lambda v, w \rangle_V = \langle T(\lambda v), T(w) \rangle_W$$

$$= \langle \lambda T(v), T(w) \rangle_W$$

$$= \lambda \langle T(v), T(w) \rangle_W$$

$$= \lambda \langle v, w \rangle_V \quad \checkmark$$

Axiom 4: Is $\langle v, v \rangle_V \geq 0$?

well, let's see!

$$\langle v, v \rangle_V = \langle T(v), T(v) \rangle_W \geq 0 \text{ since } \langle, \rangle_W \text{ satisfies 4}$$

what if $\langle v, v \rangle_V = 0$? want to know this means $v = 0$.

well, if $\langle v, v \rangle_V = 0$ then $\langle T(v), T(v) \rangle_W = 0$

But $\langle, \rangle_W$ satisfies this, so

$$\langle T(v), T(v) \rangle_W = 0 \Rightarrow T(v) = 0.$$

But T is 1-1 and as we noted in lecture,

this means $\ker T = 0$. so $T(v) = 0$

therefore means $v = 0$ ✓