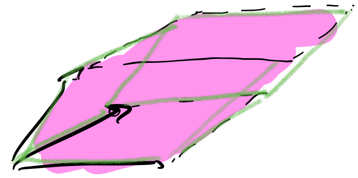
Idea of the determinant

$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

$\det A$ = signed area of 

signed means:
if axes are in standard order > 0
each swap switches sign.

or if $A \to$ lin. trans T then $\det A = \pm \dfrac{\text{volume}(T(\text{shape}))}{\text{volume}(\text{shape})}$ any shape

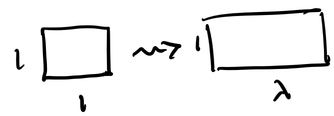
<u>consequence:</u> $\det AB = \det A \det B$

<u>intuitively:</u> $\dfrac{\text{volume}(TS(\text{shape}))}{\text{vol}(\text{shape})} = \dfrac{\text{vol}(TS(\text{shape}))}{\text{vol}(S(\text{shape}))} \dfrac{\text{vol}(S(\text{shape}))}{\text{vol}(\text{shape})}$

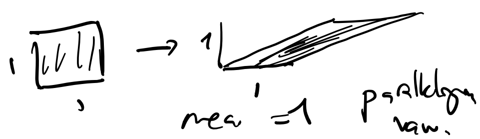
$= \det A \det B$

Some properties:

- $\det I_n = 1$
- $\det(\text{rescale entry by } \lambda) = \lambda$



- $\det(\text{swap}) = -1$
- $\det(\text{add mult. of one row to another}) = 1$



ex: $\det\begin{pmatrix}1&2\\3&1\end{pmatrix}$

$$\begin{matrix}1&2\\3&1\end{matrix} \underset{\det 1}{\rightsquigarrow} \begin{matrix}1&2\\0&-5\end{matrix} \underset{\det}{\xrightarrow{-1/5}} \begin{matrix}1&2\\0&1\end{matrix} \underset{\det 1}{\rightsquigarrow} \begin{matrix}1&0\\0&1\end{matrix}$$

So $\det\begin{pmatrix}1&2\\3&1\end{pmatrix} \det E_1 \det E_2 \det E_3 = \det I = 1$

$\det\begin{pmatrix}1&2\\3&1\end{pmatrix}\cdot\left(-\frac{1}{5}\right) = 1$

$\det = -5$

$$\det\begin{pmatrix}2&4&2\\0&5&1\\-1&2&\cancel{6}\end{pmatrix}$$

1

$$\Big\downarrow \tfrac{1}{2}$$

$$\begin{pmatrix}1&2&1\\0&5&1\\-1&2&6\end{pmatrix}$$

1

$$\Big\downarrow$$

$$\begin{pmatrix}1&2&1\\0&5&1\\0&4&7\end{pmatrix}$$

2

$$\xrightarrow{1/5}$$

$$\begin{pmatrix}1&2&1\\0&1&1/5\\0&4&7\end{pmatrix}$$

2

$$\rightarrow$$

$$\begin{pmatrix}1&2&1\\0&1&1/5\\0&0&7-4/5\end{pmatrix}$$

2

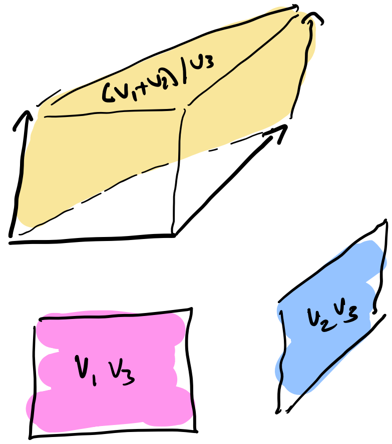
$$\begin{pmatrix}1&2&1\\0&1&1/5\\0&0&6/5\end{pmatrix}$$

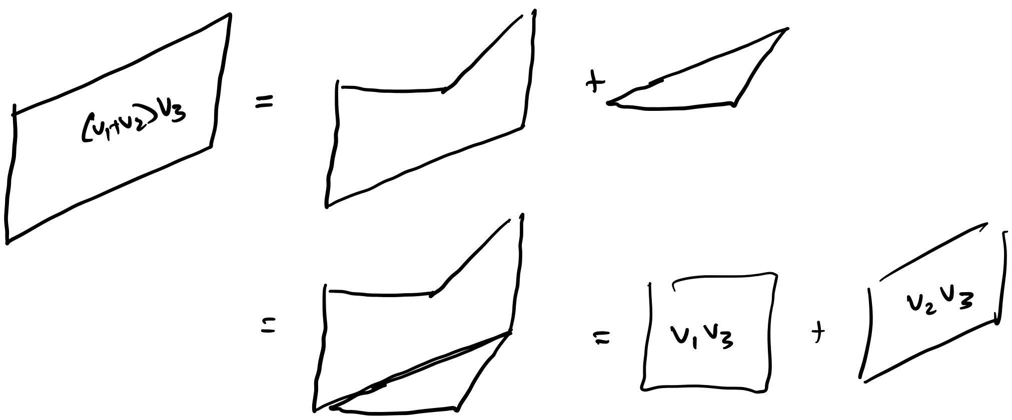
5/6

$$2\cdot 5\cdot 6/5 = \boxed{12}$$

In fact: $\det\left[v_1+v_1' \mid v_2 \mid \cdots \mid v_n\right] = \det\left[v_1 \mid \cdots \mid v_n\right] + \det\left[v_1' \mid v_2 \mid \cdots \mid v_n\right]$

in $\mathbb{R}^2$





So:

$$\det\begin{pmatrix} 2 & 4 & 2 \\ 0 & 5 & 1 \\ -1 & 2 & 1 \end{pmatrix} = \det\begin{pmatrix} 2 & 4 & 2 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$+ \det\begin{pmatrix} 0 & 4 & 2 \\ 0 & 5 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} 2 & 4 & 2 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{pmatrix} = \det\begin{pmatrix} 2 & 4 & 2 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \det\begin{pmatrix} 2 & 4 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \underbrace{\det\begin{pmatrix} 2 & 4 & 2 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{10} - \underbrace{\det\begin{pmatrix} 2 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{4}$$

etc.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \det\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$$
$$= \det\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \det\begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$
$$= \det\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \cancelto{0}{\det\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}}$$
$$- \det\begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} - \cancelto{0}{\det\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}}$$
$$= \boxed{ad - bc}$$

Alternate definition

$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is the unique function s.t.

- $\det I_n = 1$
- $\det(\text{swap two}) = -1$
- $\det(v_1 + v_1' | v_2 \cdots | v_n) = \det(v_1 | v_2 \cdots | v_n) + \det(v_1' | \cdots | v_n)$
- $\det(\lambda v_1 | v_2 | \cdots | v_n) = \lambda \det(v_1 | \cdots | v_n)$

It has the properties that

- $\det AB = \det A \det B$
- $\det A \neq 0 \iff \operatorname{rank} A = n \iff A$ is invertible

But also A invertible $\iff$ columns of A are indep.

Because $\sum a_i v_i = 0 \iff \sum a_i T(e_i) = 0$

$$\sum a_i T(e_i) = T\left(\sum a_i e_i\right)$$

$$\iff T^{-1} T\left(\sum a_i e_i\right) = 0$$

$$\iff \sum a_i e_i = 0 \iff a_i = 0 \text{ all } i.$$

So: get a test for independence

det $\neq 0$ iff columns independent (iff basis)

So:

$$\det\begin{pmatrix} 2 & 4 & 2 \\ 0 & 5 & 1 \\ -1 & 2 & 1 \end{pmatrix} = 12 \text{ tells us that}$$

$(2,0,-1)$, $(4,5,2)$, $(2,1,1)$ are independent

& are a basis.

The mirror world

$$\underset{1\times m}{[x_1 \cdots x_m]} \underset{m\times n}{\begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}} = \overset{1\times n}{[x\cdot A_1 \quad x\cdot A_2 \quad \cdots \quad x\cdot A_n]}$$

also n eqns in unknowns

from this perspective

can do column ops on right.

but - same matrices!

Wikipedia

2x3 matrix

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix} : \mathbb{R}^3 \to \mathbb{R}^2$$

$(2\times 3)\cdot(3\times 1) = 2\times 1$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ swaps 2 rows on left or 2 columns on right

$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ mults 2nd row by λ on left
col . . . right

$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ $\begin{matrix} R_1 \\ R_2 \end{matrix} \longmapsto \begin{matrix} R_1 \\ \lambda R_1 + R_2 \end{matrix}$ on left

$C_1 | C_2 \longmapsto C_1 + \lambda C_2 \mid C_2$ on right.

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ex: $\underset{A}{\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}} \xrightarrow[\underset{E_1}{\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}}]{} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \xrightarrow[\underset{E_2}{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}]{} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

2nd to 1st, $\underset{E_3}{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$E_3 E_2 E_1 A = I_2$

$A = E_1^{-1} E_2^{-1} E_3^{-1}$

$A E_3 E_2 E_1 = I_2$??

check: $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \xrightarrow[E_3]{} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow[E_2]{} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{E_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1st to 2nd !

Definition transpose. $(AB)^t = B^t A^t$

observe: transpose of row op = corresp. column op

$\Longrightarrow \det E = \det E^t$ for E elementary

$\Longrightarrow \det A = \det A^t$
clear if A is invertible.

in general) $E_1 E_2 \cdots E_r A = \mathcal{R}$ red. echelon

if $\mathcal{R}$ has 0's, get $\det A = 0$

but $\mathcal{R} = \mathcal{R}^t$

$A^t E_r^t \cdots E_1^t = \mathcal{R}^t = \mathcal{R}$

So $\det A = 0 \Leftrightarrow \det \mathcal{R} = 0 \Leftrightarrow$
$\det A^t = 0$.

$$\underset{A}{\begin{pmatrix} A_1 \\ \hline A_2 \\ \vdots \\ A_n \end{pmatrix}} \underset{B}{\begin{pmatrix} B_1 & \cdots & B_m \end{pmatrix}} = \underset{AB}{\begin{pmatrix} A_1 \cdot B_1 & A_1 \cdot B_2 & \cdots & A_1 B_m \\ \vdots & & & \\ A_n \cdot B_1 & - & \cdot & A_n \cdot B_m \end{pmatrix}}$$

$$\Big\downarrow t$$

$$\begin{pmatrix} A_1 \cdot B_1 & A_2 \cdot B_1 & \cdots & A_n \cdot B_1 \\ \vdots & & & \\ A_1 B_m & - & - & A_n \cdot B_m \end{pmatrix}$$

$$\begin{pmatrix} B_1^t \\ \hline B_2^t \\ \vdots \\ B_m^t \end{pmatrix} \begin{pmatrix} A_1^t & \cdots & A_n^t \end{pmatrix} = \begin{pmatrix} B_1^t \cdot A_1^t & \cdots & B_1^t \cdot A_n^t \\ \vdots & & \vdots \\ B_m^t \cdot A_1^t & \cdots & B_m^t A_n^t \end{pmatrix}$$

$$B^t \qquad A^t \qquad\qquad (AB)^t$$

main point

$$[v_1 \cdots v_n] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = [w_1 \cdots w_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$v^t w = w^t v$$