

(the classic Monty Hall problem with a variation)

In a game show, there are three doors. Behind one of the doors is a prize, and behind the other two are goats (not considered prizes in the context of this problem).

The contestant starts by picking one of the doors to open, but before revealing what is behind the door, the host opens one of the other doors, revealing a goat! They then ask the contestant "do you want to change your guess now?"

Should the contestant change their guess?

What if the host didn't actually know which door the prize was behind, and it just happened to reveal a goat and not a prize? Does that change anything?

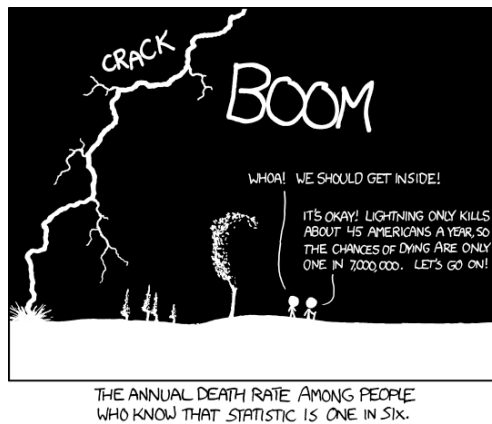
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Quick recap of determinants

Abstract vector spaces (i.e. inner products)

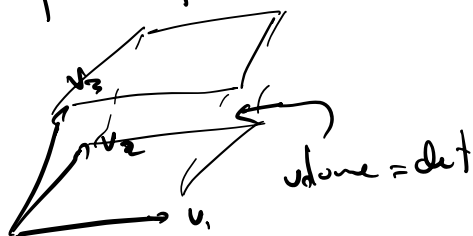
Recap: Given an $n \times n$ matrix A

$\det A$ is a number w/ following properties

- $\det(AB) = \det A \det B$

- $\det(A)$ = "signed" area of a parallelepiped spanned by columns of A

$$A = (v_1 | v_2 | \dots | v_n)$$



- $\det(v_1 + v'_1 | v_2 | \dots | v_n)$

$$= \det(v_1 | v_2 | \dots | v_n) + \det(v'_1 | v_2 | \dots | v_n)$$

- $\det(\lambda v_1 | v_2 | \dots | v_n) = \lambda \det(v_1 | \dots | v_n)$

Obtained determinant of elementary matrices

$$\det \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = 1 \quad \det \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} = \lambda \quad \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

ex: $\det \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{E_1} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{E_2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{E_2} \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \xrightarrow{E_4} \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{E_5} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = E_5^{-1} E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1}$$

$$\det A = (\det E_5)^{-1} \dots \dots \dots 1$$

$$= 1 \cdot (-1/2)^{-1} \cdot (1/2)^{-1} (-1)^{-1} = (-2)(2)(-1) = 4$$

$$\det \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = \det \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} + \det \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$$

$$= \det \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} + \det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$+ \det \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\det \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = 2 \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -2 \quad \leftarrow \text{(elem. mult.)}$$

$$\det \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = 0 \quad \text{(sized area)}$$

$$\det \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} = \det \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 + 0 \quad \text{(sized area)}$$

$$\text{ex: } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$$

$$= \det \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} + \det \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

$$+ \det \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

$$\text{ex } \det \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

$$= \det \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = 0 + 0 = 0$$

$$\begin{aligned}
 &= \det \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \\
 &= a \det \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} + b \det \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \\
 &= a d \underbrace{\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_1 + b c \underbrace{\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{-1} \\
 &= ad - bc.
 \end{aligned}$$

Reminders: one important fact: A an $n \times n$ matrix
 $\det A \neq 0 \Leftrightarrow A$ is invertible $\Leftrightarrow \text{rank } A = n$

Definition An (abstract, real) vector space is

a set V with operations $+$ and $\cdot$ such that:

$$\begin{aligned}
 + &: V \times V \rightarrow V \\
 \cdot &: \mathbb{R} \times V \rightarrow V
 \end{aligned}$$

$$\begin{aligned}
 v, w &\in V \\
 v + w &\in V
 \end{aligned}$$

$$\begin{aligned}
 \lambda &\in \mathbb{R} \\
 \lambda v &\in V
 \end{aligned}$$

$$\begin{aligned}
 +(v, w) &= v + w \\
 \uparrow \\
 &\text{"syntactic sugar"}
 \end{aligned}$$

$$\cdot(\lambda, v) = \lambda v$$

and a distinguished vector $0 \in V$
such that: $\forall v, w \in V, \lambda, \mu \in \mathbb{R}$

$$\cdot (v + w) + u = v + (w + u)$$

$$\cdot v + w = w + v$$

$$\cdot 1 \cdot v = v$$

$$\cdot 0 + v = v$$

$$\cdot (\lambda + \mu) v = \lambda v + \mu v$$

$$\cdot \lambda (v + w) = \lambda v + \lambda w$$

$$\cdot (\lambda \mu) v = \lambda (\mu v)$$

$$\cdot 0 \cdot v = 0 \in V$$

Notation:

If A, B are sets, $A \times B$ is the set of ordered pairs (a, b)

ex: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

$A \subseteq B$ subset $a \in A$ element of set

$A \times B$

Def If V is a vector space, $W \subseteq V$ a subset is called a subspace if W becomes a vector space by the operations induced by V .

i.e. $\forall w_1, w_2 \in W$ then $w_1 + w_2 \in W$; $\lambda w \in W$
if $w \in W, \lambda \in \mathbb{R}$.

ex If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ a linear transformation

then $\ker T = \{v \in \mathbb{R}^n \mid T(v) = 0\}$ is a subspace of $\mathbb{R}^n$

and $\text{im } T = \{T(v) \in \mathbb{R}^m \mid v \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$