

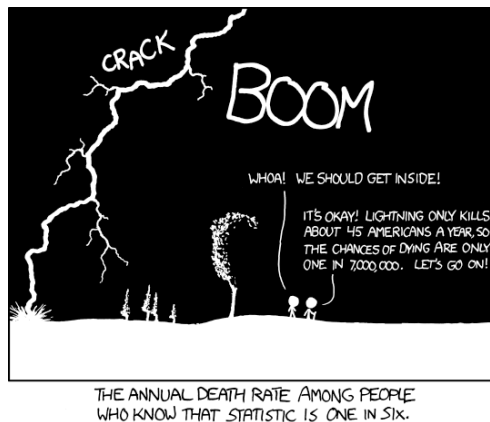
(the classic Monty Hall problem with a variation)

In a game show, there are three doors. Behind one of the doors is a prize, and behind the other two are goats (not considered prizes in the context of this problem).

The contestant starts by picking one of the doors to open, but before revealing what is behind the door, the host opens one of the other doors, revealing a goat! They then ask the contestant "do you want to change your guess now?"

Should the contestant change their guess?

What if the host didn't actually know which door the prize was behind, and it just happened to reveal a goat and not a prize? Does that change anything?



Observation

2 lectures ago we showed $\det A = \det A^T$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \end{bmatrix}$$

Last time Abstract vector spaces, subspaces

Def A (real abstract) vector space V is a set w/ operations

$$+ : V \times V \rightarrow V \quad \cdot : \mathbb{R} \times V \rightarrow V \quad \text{and a distinguished element } 0 \in V$$

$$+(v, w) = v + w \quad \cdot (\lambda, v) = \lambda v$$

such that:

$$\bullet (v + w) + u = v + (w + u)$$

$$\bullet v + w = w + v$$

$$\bullet 0 + v = v$$

$$\bullet 1 \cdot v = v$$

$$\bullet \lambda(v + w) = \lambda v + \lambda w$$

$$\bullet (\lambda + \mu)v = \lambda v + \mu v$$

$$\bullet (\lambda \mu)v = \lambda(\mu v)$$

$$\bullet 0 \cdot v = 0 \text{ all } v.$$

If $W \subseteq V$ a subset of a vector space, we say that W is a subspace (linear subspace) if for any $w_1, w_2 \in W$, $\lambda \in \mathbb{R}$, $w_1 + w_2 \in W$ and $\lambda w_1 \in W$

Def if V is a vector space, $v_1, \dots, v_n \in V$, we say $v_1, v_2, \dots, v_n$ are (linearly) independent if whenever $a_1, \dots, a_n \in \mathbb{R}$ and $a_1 v_1 + \dots + a_n v_n = 0$ we must have $a_i = 0$ all i .

Def if V is a vector space, $v_1, \dots, v_n \in V$, we define $\text{Span}\{v_1, \dots, v_n\} = \langle v_1, \dots, v_n \rangle$ to be the set of all vectors of the form $a_1 v_1 + \dots + a_n v_n$ $a_i \in \mathbb{R}$.

Ex if $V = \mathbb{R}^m$ $v_1, \dots, v_n \in \mathbb{R}^m$ $v_1, \dots, v_n$ are independent $\Leftrightarrow$ solns to $a_1 v_1 + \dots + a_n v_n = 0$ in a 's are only $a_i = 0$ all i .

$$\begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + a_2 \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + \dots + a_n \begin{bmatrix} | \\ v_n \\ | \end{bmatrix}$$

$$v_1, \dots, v_n \text{ indep.} \Leftrightarrow \text{rk} \begin{pmatrix} | & \dots & | \\ v_1 & \dots & v_n \\ | & \dots & | \end{pmatrix} = n$$

Recall: If A a matrix w/ T corresp. lin. trans.

$$\text{then } \text{null}(A) = \{v \mid Av = 0\} \quad \ker(T) = \{v \mid T(v) = 0\}$$

$$\text{If } T = \text{lin. trans for } [v_1 \mid \dots \mid v_n] \text{ then } \text{im}(T) = \langle v_1, \dots, v_n \rangle$$

Ex: consider $W = \langle \sin x, \cos x \rangle \subseteq \text{Cont}(\mathbb{R}, \mathbb{R})$
 $\{a \sin x + b \cos x \mid a, b \in \mathbb{R}\}$

$$W' = \langle 1, \sin^2 x, \cos^2 x \rangle \subseteq \text{Cont}(\mathbb{R}, \mathbb{R})$$

↑
not independent

Def $v_1, \dots, v_n$ is a basis for V if $v_1, \dots, v_n$ are independent &

$$\langle v_1, \dots, v_n \rangle = V$$

If V has a finite basis we say V is finite dimensional &
elements in basis = "dimension of V " = $\dim V$

Def If V and W are vector spaces, a function $T: V \rightarrow W$ is called a linear transformation if

$$\bullet T(v+w) = T(v) + T(w)$$

$$\bullet T(\lambda v) = \lambda T(v)$$

In particular:

$$T(0) = 0 \quad \text{because} \quad T(0) = T(0+0) = T(0) + T(0)$$

$$\begin{aligned} T(0) + (-1)T(0) &= (T(0) + T(0)) + (-1)T(0) \\ &= T(0) + (T(0) + (-1)T(0)) \end{aligned}$$

$$1 \cdot T(0) + (-1) T(0) = 0$$

$$(1 + (-1)) T(0) = 0 \cdot T(0) = 0$$

$$0 = T(0)$$

Def: If $T: V \rightarrow W$ is a linear transformation
we define $\ker T = \{v \in V \mid T(v) = 0\}$

$$\text{im } T = \{T(v) \mid v \in V\}$$

Def We say that $f: V \rightarrow W$ is an isomorphism if it is
a lin transformation & $\exists$ another lin trans. $f^{-1}: W \rightarrow V$
s.t. $f \circ f^{-1} = id_W$ $f^{-1} \circ f = id_V$

ex $\langle \cos x, \sin x \rangle = \{a \cos x + b \sin x\} \subseteq \text{Cont}(\mathbb{R}, \mathbb{R})$

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \langle \cos x, \sin x \rangle \\ a, b &\longmapsto a \cos x + b \sin x \\ f(0), f(\pi/2) &\longleftrightarrow f(x) \end{aligned}$$

$$V = \mathbb{R}^2 \quad W = \langle \cos x, \sin x \rangle$$
$$T: V \rightarrow W$$
$$T(a, b) = a \cos x + b \sin x$$

Notation: if X is a set.
 id_X is the function $X \rightarrow X$
 defined by $\text{id}_X(a) = a$ all $a \in X$

$$S: \mathbb{N} \rightarrow V$$

Remark (Vocabulary)

If $f: X \rightarrow Y$ is a function between sets X, Y we say

- f is 1-1 if $f(x) = f(x') \Rightarrow x = x'$ "horiz. line test"
or "injective" "monic"

- f is called onto if for all $y \in Y$ there exists an $x \in X$ such that $f(x) = y$
or "surjective" "epic"

- f is 1-1 & onto if and only if it has an inverse function f^{-1}
s.t. $f \circ f^{-1} = \text{id}_Y$ $f^{-1} \circ f = \text{id}_X$

$$f(f^{-1}(y)) = y \quad f^{-1}(f(x)) = x$$

Prop: Suppose $T: V \rightarrow W$ is a linear transformation
then T is injective if and only if $\ker T = \{0\}$

[10u more about this Monday]

Inner products

If V is a vector space, an inner product on V is a map

$$V \times V \rightarrow \mathbb{R} \text{ (written as } \langle v, w \rangle \text{)} \\ \text{(or } v \cdot w \text{)}$$

such that

1. $\langle v, w \rangle = \langle w, v \rangle$

2. $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$

$$3. \langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

$$4. \langle v, v \rangle \geq 0 \text{ and } \langle v, v \rangle = 0 \text{ only when } v=0. \\ (\text{positive definite})$$

$$\text{we will often write: } \|v\| = \sqrt{\langle v, v \rangle}$$

Def we say v, w are orthogonal if $\langle v, w \rangle = 0$

and a set of vectors $v_1, \dots, v_n$ is orthogonal if $\langle v_i, v_j \rangle = 0$
if $i \neq j$
and orthonormal if also $\|v_i\| = 1$.

Works same as for $\cdot$ product
i.e. same Gram-Schmidt etc.

ex: $W = \langle \cos x, \sin x, 1 \rangle$

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$$

what's the angle between
 $\cos x$ & $\sin x$?

$$v \cdot w = \|v\| \|w\| \cos \theta$$

$$\langle \cos x, \sin x \rangle = 0$$

$$\begin{array}{cc} \|\cos x\| & \|\sin x\| \\ \sqrt{\pi} & \sqrt{\pi} \end{array}$$