

Estimated (hopeful) exam grading time: late next week

This week

office hours / coffee hour 9:30-10:30 tomorrow (thurs)

next week 10:30-11:30 tuesday

Homework next week: due thursday

Rank / Nullity conversion

Lemma: if U is an invertible $m \times m$ matrix, A an $m \times n$ matrix then $\text{image of } A$ is isomorphic to $\text{image of } UA$.

$$A \longleftrightarrow \text{lin. trans } T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\text{im } A \equiv \text{im}(T) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

$$\text{im } UA = \{UAx \mid x \in \mathbb{R}^n\} \quad \begin{matrix} \uparrow \\ \text{subspace} \end{matrix}$$

Claim of lemma: there exists a bijective linear transformation

$$\text{im } A \longrightarrow \text{im } UA$$

$$y \longmapsto Uy$$

$$\begin{matrix} \text{"} \\ Ax \end{matrix} \qquad \begin{matrix} \text{"} \\ UAx \end{matrix}$$

$$\text{im } A \longleftarrow \text{im } UA$$

$$Ax = U^{-1}UAx = U^{-1}z \longleftarrow z = UAx$$

Aside (about functions)

if $f: X \rightarrow Y$ is a function between two sets X & Y
then f is bijective if and only if there is an inverse function
 $f^{-1}: Y \rightarrow X$ s.t. $f \circ f^{-1} = \text{id}_Y$ $f^{-1} \circ f = \text{id}_X$

Lemma: If U is invertible $m \times m$, A an $m \times n$ matrix then
 $\ker A = \ker UA$.

$$A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Pf idea

$$x \in \ker A \Leftrightarrow Ax = 0$$

$$\Updownarrow$$

$$x \in \ker UA \Leftrightarrow UAx = 0$$

$$UA: \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \xrightarrow{U} \mathbb{R}^m$$

Lemma: if A is $m \times n$, U is an inv $n \times n$ then

$$\text{im } AU = \text{im } A$$

$$\mathbb{R}^n \xrightarrow{U} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

$\searrow$
 AU

Lemma: if A is $m \times n$, U is an inv $n \times n$ then

$$\ker AU \cong \ker A$$

Upshot: dimension of image of A is unchanged by row ops!
dimension of $\ker$ of A is unchanged by column ops!
nullity of A .

Row ops $\rightsquigarrow$

$$\begin{bmatrix} 1 & \times & - & \text{---} \\ 0 & 0 & 1 & - \\ 0 & 0 & 0 & 1 \\ 0 & - & - & 0 \end{bmatrix}$$

column ops



image = span of columns

rank A = # of nonzero rows in RREF = # of nonzero columns in RREF = dim of image of A .

kernel A = vectors $\perp$ to each row

$$= \{ \text{vectors w/ first } \overset{\text{rank } A}{r} \text{ coords} = 0 \} \subseteq \mathbb{R}^n$$

has basis

$$\begin{bmatrix} 0 & \text{---} & 0 & 1 & 0 & \text{---} \\ 0 & \text{---} & 0 & 0 & 1 & \text{---} \end{bmatrix}$$

there are $n-r$ of these.

Theorem "rank nullity theorem" Fundamental theorem of linear algebra
if A is an $n \times n$ matrix then $\boxed{\text{rank } A + \text{nullity } A = n}$

Bases & Matrix representations of linear transformations

Let V be a vector space

Recall: If $B = (v_1, \dots, v_n)$ is a list of vectors in V
then we say B is a basis if

B is max'd independent
or, equivalently

B is min'd spanning

we say in this case that $\dim V = n$.

If $\dim V = n$ then any set of n indep. vectors is a basis
& any set of n spanning vectors is a basis

Note: if A is an $n \times n$ matrix $\Leftrightarrow \mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$

then $\text{im } A = \mathbb{R}^n \Leftrightarrow$ columns of A are spanning $\mathbb{R}^n$
 $\Leftrightarrow$ columns are a basis for $\mathbb{R}^n$

$\text{rank } A = n \Leftrightarrow \text{nullity } A = 0$

$\swarrow$ surj

$\nwarrow$ injective

$\Leftrightarrow A$ is invertible

$\Leftrightarrow \det A \neq 0$

$\Leftrightarrow \det A^T \neq 0$

If V is a vector space & $B = (v_1, \dots, v_n)$ is a basis for V

then we can define a linear trans

$$T_B = T: \mathbb{R}^n \longrightarrow V$$

$$e_i \longmapsto v_i$$

$$T(e_i) = v_i$$

this is injective since $\ker = 0$ since v_i 's are indep.

$$T\left(\sum_{\substack{\text{in } \mathbb{R}^n}} c_i e_i\right) = 0 \Rightarrow T\left(\sum c_i e_i\right) = 0$$

$$= \sum T(c_i e_i)$$

$$= \sum c_i T(e_i)$$

$$= \sum c_i v_i \text{ indep}$$

$$\Rightarrow c_i = 0 \text{ all } i.$$

$$\Rightarrow \sum c_i e_i = 0.$$

T is surjective since v_i 's are in $\text{im } T$ & span V !

$\Rightarrow T$ is an isomorphism.

If $T: V \longrightarrow W$ a linear transformation

how can we express T in terms of a matrix?

if $B = (v_1, \dots, v_n)$
basis for V

$C = (w_1, \dots, w_m)$
basis for W

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{T_B^{-1} T T_B} & \mathbb{R}^m \\ T_B \downarrow & \curvearrowright & \downarrow T_C \\ V & \xrightarrow{T} & W \end{array}$$

$$\uparrow T_C^{-1}$$

Def the matrix corresp. to $T|_{\mathcal{L}}: \mathcal{L} \rightarrow \mathcal{L}$ above is called the matrix for T w.r.t to the bases $\mathcal{B}$ & $\mathcal{L}$ and is written as $P_{\mathcal{L}}^{\mathcal{B}}$

ex: $P_2 = \text{polys of deg} \leq 2$ $P_1 = \text{polys of deg} \leq 1$

$$\frac{d}{dx}: P_2 \rightarrow P_1$$

$$\mathcal{L} = (1, x)$$

$\mathcal{B}$ for P_2

"
(1, x, x^2)

$$P_2 \rightarrow P_1$$

$$\begin{matrix} 1 & x & x^2 \\ \uparrow & \uparrow & \uparrow \\ e_1 & e_2 & e_3 \end{matrix} \quad \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} T_{\mathcal{L}}$$

$$\begin{matrix} 1 & x \\ \uparrow & \uparrow \\ e_1 & e_2 \end{matrix} \quad \begin{matrix} \uparrow \\ \uparrow \end{matrix} T_{\mathcal{L}}$$

$$P_{\mathcal{L}}^{\frac{d}{dx}} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

where e_1 goes

where e_2 goes

where e_3 goes

$$e_1 \mapsto 0$$

$$e_2 \mapsto e_1 \quad \begin{matrix} \text{via } x \\ \text{via } d/dx \end{matrix}$$

$$e_3 \mapsto 2e_2 \quad \begin{matrix} \text{via } x^2 \\ \text{via } d/dx \end{matrix}$$

ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{array}{ccc} e_1 & e_2 & \mathbb{R}^2 \\ \uparrow & \uparrow & \uparrow T_B \\ e_1 & e_2 & \mathbb{R}^2 \end{array} \xrightarrow{T} \begin{array}{ccc} \mathbb{R}^2 & (1,1) & (0,2) \\ \uparrow T_e & \uparrow & \uparrow \\ \mathbb{R}^2 & e_1 & e_2 \end{array}$$

basis $B = ((1,0), (0,1))$ $\mathcal{C} = ((1,1), (0,2))$

$$\mathcal{C}^T_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^T_B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

In practice we think about $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$T: \mathbb{R}^1 \hookrightarrow \mathbb{R}^1$$

it's more natural to choose one basis for $\mathbb{R}^1$

and write B^T_B

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad B = ((1,-1), (0,1))$$

$$T((1,-1)) = T((1,0) + (0,-1))$$

$$= T(e_1 - e_2) = T(e_1) - T(e_2)$$

$$= (1,1) - (0,2) = (1,-1)$$

$$T((0,1)) = T(e_2) = 2e_2 = 2(0,1)$$

$$B^T_B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$