

Matrices for linear transformations

$T: V \rightarrow W$ linear transformation.

$B = (v_1, \dots, v_n)$ basis for V $C = (w_1, \dots, w_m)$ basis for W

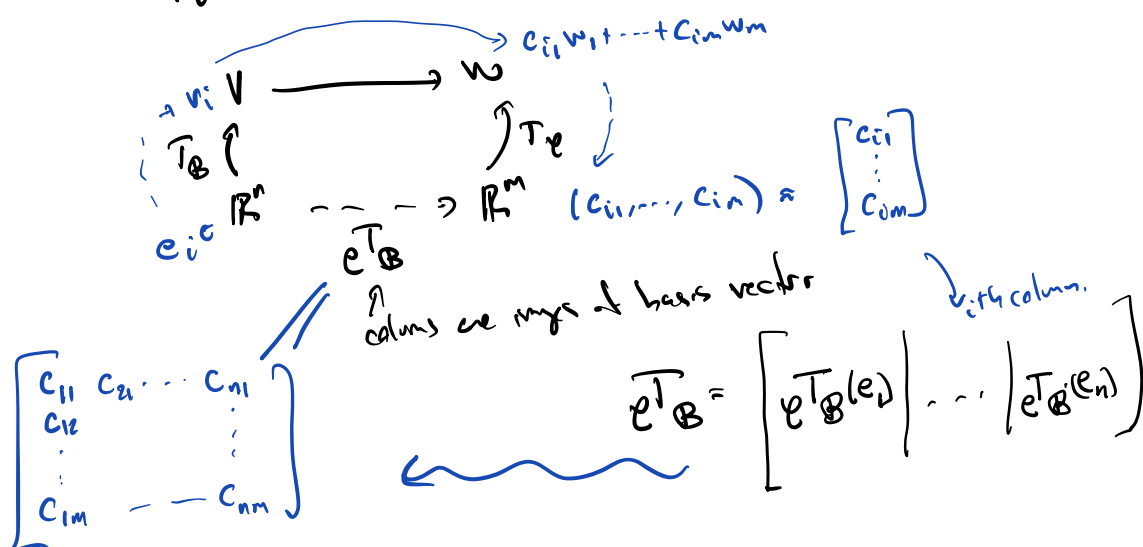
then B lets us identify $V \cong \mathbb{R}^n$
 C — — — $W \cong \mathbb{R}^m$

$v \in V$ can write it uniquely as $v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n \in V$
 $(a_1, \dots, a_n) \in \mathbb{R}^n \xrightarrow{T_B}$

can then think about $T: V \rightarrow W$
 $\parallel$ $\parallel$
 $\mathbb{R}^n \rightarrow \mathbb{R}^m$
 $\uparrow$ given a matrix $C^T B$

$T(v_i) \in W$ has basis $C = (w_1, \dots, w_m)$

$$T(v_i) = c_{i1} w_1 + c_{i2} w_2 + \dots + c_{im} w_m$$



Punchline ^{ith} column = how to write T (with many basis vecs) in terms of the outgoing basis.

ex: $P_2 \xrightarrow{T} \mathbb{R}^3$ polys. $\deg \leq 2 \xrightarrow{2/x} \text{poly } \deg \leq 1$
 $f \mapsto (f(1), f(2), f(3))$

$\mathcal{B}$ for P_2 $\mathcal{B} = (1, x, x^2)$ $\mathcal{C}$ for $\mathbb{R}^3$ $\mathcal{C} = (e_1, e_2, e_3)$

$${}_{\mathcal{C}}T_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

$$T(1) = (1, 1, 1) = 1e_1 + 1e_2 + 1e_3 \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T(x) = (1, 2, 3) = 1e_1 + 2e_2 + 3e_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T(x^2) = (1, 4, 9) = 1e_1 + 4e_2 + 9e_3 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

$T: V \rightarrow V$ it's more natural to pick a single basis for domain & range.

Focus on $\mathcal{B} \mapsto {}_{\mathcal{B}}T_{\mathcal{B}}$

Life is easier when we can find a basis for a lin. trans. which makes the matrix ${}_{\mathcal{B}}T_{\mathcal{B}}$ diagonal

i.e. of form $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$

i.e. in this basis $\mathcal{B} = (v_1, \dots, v_n)$ we have $T(v_i) = \lambda_i v_i$

lots of the time, can find such bases.
 most of the time that we can, get λ_i 's are distinct.

Def: if $T: V \rightarrow V$ is a l.w. transformation
 $\lambda \in \mathbb{R}$, $v \in V$, we say v is an eigenvector for T w/ eigenvalue λ
 if $Tv = \lambda v$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$Ae_1 = e_1 \quad Ae_2 = 2e_2$$

$$\begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \stackrel{?}{=} \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 = \lambda x_1$$

$$2x_2 = \lambda x_2$$

$$\text{if } x_1 \neq 0 \Rightarrow \lambda = 1$$

$$x_2 \neq 0 \Rightarrow \lambda = 2$$

$$\Rightarrow x_1 \neq 0 \Rightarrow v = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 e_1$$

$$x_2 \neq 0 \Rightarrow v = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 e_2$$

Remark:

The following are equivalent for $T: V \rightarrow V$ a l.w. trans.

- There exists a basis $\mathcal{B}$ for T consisting of eigenvectors for T
- There exists a basis $\mathcal{B}$ for T such that $T_{\mathcal{B}\mathcal{B}}$ is diagonal.

ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ matrix $\begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix} = A$

1 an eigenvalue?

$$\begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} 4x_1 + 4x_2 = x_1 \\ 7x_1 + 16x_2 = x_2 \end{cases} \rightsquigarrow \begin{cases} 3x_1 + 4x_2 = 0 \\ 7x_1 + 15x_2 = 0 \end{cases} \quad \begin{bmatrix} 3 & 4 \\ 7 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 4 \\ 7 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightsquigarrow x_1 = 0 = x_2 \overset{!}{\text{is}} \text{ not an eval.}$$

$$\begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

find λ an eval?

$$\begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4-\lambda & 4 \\ 7 & 16-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \text{non-trivial?}$$

$$\text{if } \begin{bmatrix} 4-\lambda & 4 \\ 7 & 10-\lambda \end{bmatrix} \text{ invertible} \Rightarrow \text{no solns except } x=0$$

$$\text{if not invertible} \Rightarrow \text{rank} < 2$$

$$\Rightarrow \text{nullity} \geq 1$$

$$\Rightarrow \text{kernel} \neq 0$$

$$\Rightarrow \exists x \in \mathbb{R}^2 \text{ such that}$$

$$\begin{bmatrix} 4-\lambda & 4 \\ 7 & 10-\lambda \end{bmatrix} x = 0$$

i.e.

$$\begin{bmatrix} 4 & 4 \\ 7 & 10 \end{bmatrix} x = \lambda x$$

so $\exists$ an eigenvalue λ .

Criteria:

$$\text{if } \det(A - \lambda I_n) = 0$$

then λ is an eigenvalue.

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} a_{11}-\lambda & & \\ & \ddots & \\ & & a_{nn}-\lambda \end{bmatrix} = a_n \lambda^5 + a_4 \lambda^4 + \dots - \lambda + a_0$$

if we can find a basis of eigenvectors for $T: V \rightarrow V$

then T_B is diagonal.

not every matrix works in this way:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

if v is an eigenvector w/ e.val λ then

$$A^2 v = A A v = A \lambda v = \lambda A v = \lambda \lambda v = \lambda^2 v$$

$$A^2 v = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} v = -v \quad \lambda^2 = -1$$

$$\lambda \notin \mathbb{R}$$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \det(B - \lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 \quad \det = 0 \Rightarrow \lambda = 1$$

only allowable e.val is 1 so if have a basis

v_1, v_2 then $\Rightarrow$ that basis matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow B = I_2$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 + x_2 = x_1 \Rightarrow x_2 = 0$$

$$x_2 = \cancel{x_2}$$

Def $\det(A - I_n \lambda)$ is the
"characteristic polynomial"

Upshot: even if char poly factors into linear pieces ($n \times n$ matrix)
we need not have a basis for which we are diagonal.

Bot: (Fact) if roots of char poly are distinct then
it will have a basis of eigenvectors.