

Given $V \xrightarrow{T} W$ linear transformation between vector spaces

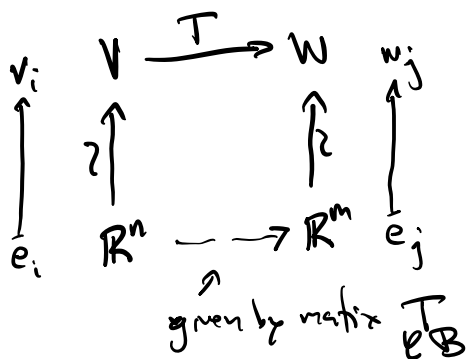
{ bases $B = (v_1, \dots, v_n)$ for V ; $C = (w_1, \dots, w_m)$ for W

get a matrix $T_B = \begin{bmatrix} [Tv_1]_C & [Tv_2]_C & \dots & [Tv_n]_C \end{bmatrix}$

$[w]_C = w$ in basis C

or $w = a_1 w_1 + a_2 w_2 + \dots + a_m w_m$

$$[w]_C = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

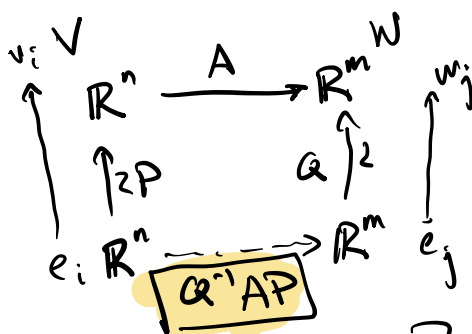


Change of basis

suppose we have
new bases B for $\mathbb{R}^n$

$(v_1, \dots, v_n)$

{ C for $\mathbb{R}^m$
" $(w_1, \dots, w_m)$



$$\text{let } P = [v_1 | \dots | v_n]$$

change of basis
matrix

$$Q = [w_1 | \dots | w_m]$$

Summary: to write A in form basis $v_1, \dots, v_n$ means $w_1, \dots, w_m$ at top

$$P = [v_1, \dots, v_n] \quad Q = [w_1, \dots, w_m]$$

A new bases is $\overline{QAP}$

ex: $A = \begin{bmatrix} 2 & 1 & 9 \\ 3 & 5 & 1 \end{bmatrix} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$P_r = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 5 & 3 & | & 0 & 1 \end{bmatrix}$$

$$\downarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -7 & -5 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 5/2 & -1/2 \end{array} \right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -3/7 & 2/7 \\ 0 & 1 & 5/7 & -1/7 \end{array} \right] \quad \underbrace{\hspace{1.5cm}}_{Q^{-1}}$$

new basis

$$Q^{-1} \begin{bmatrix} 2 & 1 & 9 \\ 3 & 5 & 1 \end{bmatrix} P$$

$$\begin{bmatrix} -3/7 & 2/7 \\ 5/7 & -1/7 \end{bmatrix} \begin{bmatrix} 2 & 1 & 9 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[[Av_1]_\varphi \mid [Av_2]_\varphi \mid [Av_3]_\varphi]$$

$$= \left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}_e \mid \begin{pmatrix} 1 \\ 5 \end{pmatrix}_e \mid \begin{pmatrix} 2+9 \\ 3+1 \\ 11 \\ 4 \end{pmatrix}_e \right] = \left[\begin{array}{ccc} 0 & 1 & 0 \end{array} \right]$$

last one is $\begin{pmatrix} 9 \\ 6 \end{pmatrix}$

where $\begin{pmatrix} a \\ b \end{pmatrix}$ solves

$$\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

Today: Diagonalization

Punchlines:

- If eigenvalues are all real & distinct, they are relatively easy to find algorithmically

(eigenvalues are solutions to $\det(A - \lambda I_n) = 0$
in λ poly-fdeg n .)

- If A is a symmetric matrix ($A = A^T$)
then all eigenvalues are real & can find a basis
consisting of eigenvectors which are orthonormal.
(spectral theorem / principal axis theorem)

How to find eigenvalues if they are real?

Imagine A
 $n \times n$

$v_1, \dots, v_n \in \mathbb{R}^n$
Basis B of eigenvectors w/ evals
 $\lambda_1 > \lambda_2 > \dots > \lambda_n$ real #s.

$$v = a_1 v_1 + \dots + a_n v_n$$

$$Av = \lambda_1 a_1 v_1 + \dots + \lambda_n a_n v_n$$

$$= A a_1 v_1 + A a_2 v_2 + \dots + A a_n v_n$$

$$a_1 \underbrace{Av_1}_{\lambda_1 v_1} + \dots + a_n \underbrace{Av_n}_{\lambda_n v_n}$$

$$\frac{A^N v}{\|A^N v\|} = \frac{\lambda_1^N a_1 v_1 + \dots + \lambda_n^N a_n v_n}{\|A^N v\|} \approx \pm v_1$$

$$A = \begin{bmatrix} 3 & 7 & 1 \\ 7 & 2 & -5 \\ 1 & -5 & 1 \end{bmatrix}$$

$$v = (1, 0, 0)$$

$$x \xrightarrow{F} \frac{Ax}{\|Ax\|} \quad F^{1000}(v)$$

$$v' = (0.63, 0.71, -0.31)$$

$$Av' = (6.557, 7.392, -3.253)$$

$$\frac{-3.253}{-0.31} = \frac{7.392}{0.71} = 10.397$$

$$\frac{6.557}{.63} = 10.397$$

Theorem Every $n \times n$ matrix has a complex eigenvector.

Proof: e. values are solutions to $\det(A - \lambda I_n) = 0$
(ch. poly.)

degree n poly. in λ

Fundamental theorem of algebra: every nonconstant poly has a root in $\mathbb{C}$.

so there exists $\lambda \in \mathbb{C}$ such that $\det(A - \lambda I_n) = 0$

$\Rightarrow$ for $A - \lambda I \neq 0$ i.e. $\exists v$ s.t.
 $(A - \lambda I)v = 0$

$$\text{ex: } A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \det(A - \lambda I) = \lambda^2 + 1$$

$$\Rightarrow Av - \lambda I_n v = 0 \\ Av - \lambda v = 0 \\ Av = \lambda v$$

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix} = i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Prop: If A is symmetric (or, more generally, Hermitian)
then A has a real eigenvector w/ real eigenvalue.
(actually all evals are real!)

(A symmetric means $A^T = A$
 A Hermitian means $A^T = \bar{A}$ = conj. each entry)

Pf: Suppose λ is an eval for A . so $Ax = \lambda x$ some x
(maybe $\lambda \in \mathbb{C}$)

$$\left[\begin{array}{l} \text{Def if } v \in \mathbb{C}^n \\ (v_1, \dots, v_n) \end{array} \quad \|v\| = \sqrt{|v_1|^2 + |v_2|^2 + \dots + |v_n|^2} \right. \\ \left. = \sqrt{v_1 \bar{v}_1 + \dots + v_n \bar{v}_n} \right]$$

$$\|v\|^2 = v_1 \bar{v}_1 + v_2 \bar{v}_2 + \dots + v_n \bar{v}_n$$

$$= v^T \bar{v} \in \mathbb{R}$$

$$\text{let } c = v^T \bar{v} \in \mathbb{R}$$

$$\lambda c = \lambda v^T \bar{v} \\ = (\lambda v)^T \bar{v} \\ = (Av)^T \bar{v}$$

$\mathbb{R}$ real #s

$\mathbb{C}$ complex #s

$$a+bi \quad a, b \in \mathbb{R}$$

$$|a+bi| = \sqrt{a^2 + b^2}$$

$$\overline{a+bi} = a-bi$$

$$(a+bi)(\overline{a+bi}) \\ = (a+bi)(a-bi) = a^2 - (bi)^2$$

$$= v^T A^T v \quad A \text{ Hermitian}$$

$$= a^2 + b^2,$$

$$= v^T \bar{A} v \quad A^T = \bar{A}$$

$$= v^T (\overline{Av})$$

$$= v^T (\overline{\lambda v}) = v^T \bar{\lambda} v = \bar{\lambda} v^T v = \bar{\lambda} c \quad \lambda = \bar{\lambda}$$

$$a+bi = a-bi$$

$$\Rightarrow \lambda \in \mathbb{R}! \quad \begin{matrix} 2bi=0 \\ b=0 \end{matrix}$$

Note: if λ is a real eigenval. of A real matrix

then $\exists$ an eigenvector ^{which} is also real.

$A - \lambda I_n$ has a kernel
(as a real
linear trans.)

Sidebar: orthogonal matrices & orth. change of basis

Def An orthogonal matrix is one of the form

$$U = [v_1 | \dots | v_n] \text{ where } (v_i) \text{ are an orthonormal basis.}$$

$$v_i \cdot v_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \Leftrightarrow U^T U = \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \dots & v_1^T v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n^T v_1 & \dots & \dots & v_n^T v_n \end{bmatrix}$$

$$\text{i.e. } U^T = U^{-1} \quad = I_n$$

or $U =$ (o.basis matrix to an orthonormal basis.

$$\begin{aligned} \|Ux\| &= \sqrt{(Ux)^T (Ux)} = \sqrt{x^T U^T U x} \\ &= \sqrt{x^T I_n x} = \sqrt{x^T x} = \|x\| \end{aligned}$$

$$u_x \cdot u_y = (u_x)^T (u_y) = x^T u^T u y = x^T y = x \cdot y$$

Theorem If A is symmetric, can find an orthonormal basis of eigenvectors.