

Last time

- Change of basis / matrices for lin transformations
- Every matrix has a complex eigenvector / eigenvalue
- If A is a symmetric matrix, it always has a real eigenvalue.
in fact: showed all eigenvalues are real.

Theorem: Spectral theorem / Principal axis theorem

If A is a symmetric matrix, we can find an orthonormal basis of eigenvectors.

H = "Hamiltonian lin. trans" encodes info about "energy"

H is diagonalizable (actally close to symmetric (Hermitian))
eigenvalues = enrg levels.

Last time: can efficiently find eigenvects by iteration of lin. transformations.

if T has e. values $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ real

Then choose $v \in \mathbb{R}^n$ randomly $\frac{Tv}{\|v\|}$ over & over.

$Tv, T^2v, \dots$ λ_1 dominate.

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

v_i vect w/ eul λ_i

$$T^N v \approx a_1 \lambda_1^N v_1$$

$$\frac{T^N v}{\|v\|} \approx v_1$$

(v_i length 1)
basis.

Warmup:

Def: a lin transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal

$$\text{if } (Tv) \cdot (Tw) = v \cdot w$$

Def: a matrix A is orthogonal if $A^T = A^{-1}$

Lemma: Suppose T is a lin trans w/ matrix A as abv.

Then the following are equivalent:

1. A is orthogonal

4. columns of A are orthonormal
bas. 3

2. T is orthogonal

5. rows of A are orthonormal
bas. 3

3. A^T is orthogonal.

Note: A orthogonal $\Leftrightarrow A^T = A^{-1} \Leftrightarrow AA^T = I_n$

A^T is orthogonal $\Leftrightarrow A^T(A^T)^T = I_n \Leftrightarrow A^T A = I_n$
so $1 \Leftrightarrow 3$

If A is orthogonal then $Tv \cdot Tw = (Av)^T (Aw)$

(1 $\Rightarrow$ 2)

$$= v^T A^T A w = v^T w = v \cdot w$$

$$Av \cdot w = (Av)^T w = v^T (A^T w) = v \cdot A^T w$$

$$A^T A = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}^T \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \dots \end{bmatrix}$$

$$A^T A = I_n \Leftrightarrow v_i \cdot v_j = 1 \text{ if } i=j \text{ or } 0 \text{ else.}$$

i.e. v_i are orthogonal
 i length 1

Side comment:

if $\det A = 1$ & A is orthogonal then A is a "rotation"

other possibility: $\det A = -1$ & these are rotations followed by reflections.

Theorem: Principal axis theorem

let A be an $n \times n$ symmetric matrix. Then we can find an orthonormal basis of eigenvectors.

Pr: By induction on n

$n=1$ 1×1 symmetric matrix $\mathbb{R}^1 \rightarrow \mathbb{R}^1$

$[\lambda]$ $e_1 =$ eigen vector
w/ eigenvalue λ

suppose true for $n-1$

let A be an $n \times n$ matrix. let x_1 be an eigen vector w/ eigenvalue λ_1

let $\{x_1, \dots, x_n\}$ be an orthonormal basis

↑
normalize x_1
so it has length 1

Gram-Schmidt. $B = \{x_1, \dots, x_n\}$

$$P^{-1}AP = {}_B A_B = \begin{bmatrix} \lambda_1 & \boxed{?} \\ 0 & \boxed{?} \\ \vdots & \vdots \\ 0 & \vdots \end{bmatrix}$$

$$P = [x_1 | \dots | x_n]$$

note: because $\{x_i\}$ are orthonormal P is orthogonal

$$\Rightarrow P^{-1} = P^T$$

$$P^T A P = P^{-1} A P = \begin{bmatrix} \lambda & \boxed{0} \\ 0 & \boxed{?} \\ 0 & \end{bmatrix} \quad \text{; } P^T A P \text{ is symmetric}$$

$$(P^T A P)^T = P^T A^T P^T = P^T A P \quad \checkmark$$

$$= \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \boxed{B} \\ 0 & \end{bmatrix} \quad \text{where } B \text{ is symmetric}$$

So can find a basis of eigenvectors for B by induction.

$y_2, y_3, \dots, y_n$ eigenvectors for B

$$\tilde{y}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ y_i \\ \vdots \end{bmatrix} \in \mathbb{R}^n$$

$$P^{-1} A P \tilde{y} \quad \text{if } \lambda \text{ is equal to } y$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ y \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \lambda y \\ \vdots \end{bmatrix} = \lambda \tilde{y}$$

So $\tilde{y}_2, \dots, \tilde{y}_n \in \mathbb{R}^n$ orthogonal to each other & to e_1

and eigenvectors of $\begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & B \end{bmatrix} = P^{-1} A P$

So $P e_1, P \tilde{y}_2, \dots, P \tilde{y}_n$ are ^{orthogonal eigenvectors} a basis for A

$P e_1 = x_1$ eigenvector.

$$A P \tilde{y}_i = P P^{-1} A P \tilde{y}_i$$

$$= P \lambda_i \tilde{y}_i = \lambda P \tilde{y}_i$$

since $\tilde{y}_i$ are eigenvectors for $P^{-1} A P$