Johnny is a free spirit. Every morning he walks to the train station & takes the first train, wherever it goes.

There are two trains that each come every hour.

One goes to Glousterberryshire & the other to Trigub.

Even though he gets there at a fairly random time between 9 & 11, Johnny notices that he ends up in Trigub roughly 80% of the time.

As the trains both come regularly, on time, once an hour, why would this be the case?

Easiest kinds of systems of eqns:

$$x = 7$$

$$\boxed{y} + 3z - w = 9$$

$$y = 9 - 3z + w$$

freely choose.

$w = t$, $z = s$ } free params

$$y = 9 - 3s + t$$

"parametrized solutions"

$x - 3y = 4 \qquad z + 4w = 9$

$x = 4 + 3y \qquad z = 9 - 4w$

$$\boxed{\begin{array}{ll} y = s & w = t \\ x = 4 + 3s & z = 9 - 4t \end{array}}$$

<u>example:</u>

$$\begin{aligned} x - 2y + 3z - 2w &= 0 \\ -3x + 6y + z &= 0 \\ -2x + 4y + 4z - 2w &= 0 \end{aligned}$$

"homogenous" = (const. term = 0)

$$\begin{array}{l} \\ +3 \\ +2 \end{array} \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$ (0's implied in hom. case)

↓ 2 row ops.

$$\begin{array}{l} \\ \\ -1 \end{array} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 0 & 10 & -6 \\ 0 & 0 & 10 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 0 & 10 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ 1/10 \\ \\ \end{array}$$

"row echelon form"

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$10z - 6w = 0$

$z - \frac{6}{10}w = 0$

"staircase pattern"

Def A matrix is in row echelon form if

each row is either 0 or starts with a 1 "leading 1's"

- each leading 1 is to right of all leading 1's above it.

Def A matrix is in reduced row echelon form if it's in row echelon form &, each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{echelon}$$

$x - 2y + 3z - 2w = 0$

$z - 3/5\, w = 0$

advantage: solve via "back substitution"

w = free parameter, z = defined

y = free, x = defined.

$w = s \quad z = 3/5\, s$

$$\begin{bmatrix} 1 & -2 & 0 & -1/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

"reduced row echelon form"

$x - 2y - 1/5\, w = 0$

$z - 3/5\, w = 0$

$$x = 2y + \tfrac{1}{5}w$$

$$z = \tfrac{3}{5}w$$

$y = s \quad w = t$

$$\boxed{x = 2s + \tfrac{1}{5}t, \quad y = s, \quad z = \tfrac{3}{5}t \quad w = t}$$

<u>Last time</u>: we described this via "row operations"

reversible process of rewriting system of eqns

- swap two rows
- multiply row by nonzero number
- add one row to another.

<u>Fact</u>: Via row op's, can always get to (reduced) row echelon form.

<u>In practice</u>:

for solving a single (large) system: use row echelon

for solving lots of equations w/ same coeffs: <u>use reduced</u>.

example:

$$-2\left(\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 5 & 2 \end{array}\right]\right. \quad \begin{array}{cc} 1 & a \\ 2 & b \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -1 & 2 \end{array}\right] \begin{array}{cc} 1 & a \\ 0 & b-2a \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & -2 \end{array}\right] \begin{array}{cc} 1 & a \\ 0 & 2a-b \end{array}$$

$y = 0$

$x + 3y = 1$

$x = -3y + 1$

$x = 1$

$y = -2$

$x + 3y = 0$

$x = -3y = -3(-2) = 6$

$$-3\left(\left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & 1 & 2a-b \end{array}\right]\right. \longleftrightarrow \begin{array}{l} x + 3y = a \\ y = 2a-b \end{array}$$

reduced

$$\left[\begin{array}{cc|c} 1 & 0 & a - 3(2a-b) \\ 0 & 1 & 2a-b \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -5a+3b \\ 0 & 1 & 2a-b \end{array}\right]$$

$\updownarrow$

$x = -5a + 3b$

$y = 2a - b$

Mention: in text → "Gaussian Algorithm"
procedure for doing a series of row ops to get to (reduced) row echelon form.

$$\begin{array}{c} \\ \tfrac{1}{3} \end{array}\begin{bmatrix} 3 & 5 & 7 & 8 \\ 2 & 3 & 1 & 0 \\ 0 & 5 & 9 & \cdot \end{bmatrix} \to \begin{bmatrix} 1 & ? & ? & ? & \cdots \\ 0 & 0 & \cdots & & \\ 0 & 2 & \cdots & & \end{bmatrix}$$

Def: the rank of a matrix = # of nonzero rows in row echelon form.

ex: $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \xrightarrow{\text{red.ech.}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ rank = 2

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix} \xrightarrow{\text{ech.}} \begin{bmatrix} 1 & -2 & 0 & -1/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 2

Homogeneous equations

nice because solutions can be combined to get new solutions:

$$3x + 2y + 5z = 0$$

$\left.\begin{array}{l} (x_1, y_1, z_1) \\ (x_0, y_0, z_0) \end{array}\right\}$ solns

$3x_0 + 2y_0 + 5z_0 = 0$ true.

$3x_1 + 2y_1 + 5z_1 = 0$

$$3(x_0+x_1)+2(y_0+y_1)+5(z_0+z_1) = 0$$

if (x_0, y_0, z_0) & (x_1, y_1, z_1) are both solns

then so is $(x_0+x_1, y_0+y_1, z_0+z_1)$

Similarly: if λ is a real number

(x_0, y_0, z_0) soln $\Rightarrow$ $(\lambda x_0, \lambda y_0, \lambda z_0)$ also a solution.

$3x_0 + 2y_0 + 5z_0 = 0$ true.

$$3(\lambda x_0) + 2(\lambda y_0) + 5(\lambda z_0) = 0$$

Definition: a tuple vector is a list of numbers $(a_1, a_2, \ldots, a_n)$

add them $(a_1, a_2, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1+b_1, \ldots, a_n+b_n)$

scalar mult. $\lambda(a_1, \ldots, a_n) = (\lambda a_1, \ldots, \lambda a_n)$

Def: A column vector is a list of numbers $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

(same kind of stuff as above)

Def A row vector is ... $[a_1 \cdots a_n]$

If we have a set S of tuple vectors
we say they are a **space** if adding two vectors in
S gives another vector in S & if
scalar multiples of vectors in S are also in S.

Proposition: if we are given a collection of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0\\
a_{21}x_1 + \cdots\cdots + a_{2n}x_n &= 0\\
&\vdots\\
a_{m1}x_1 + \cdots\cdots + a_{mn}x_n &= 0
\end{aligned}
$$

then the set of solutions forms a space (of tuples)
i.e. if $\vec{p} = (p_1, \ldots, p_n)$ & $\vec{q} = (q_1, \ldots, q_n)$ are solns
to this system, so is $\vec{p} + \vec{q}$ & so is $\lambda\vec{p}$
for $\lambda \in \mathbb{R}$.

Forms a "solution space"