Puzzler:

There is a hallway with switches labelled 1 to 10, which are all in the "off" position. 10 people pass through the hallway.

The first person flips each switch (turning them all on).

The second person flips every switch whose label is a multiple of 2 (turning them off again)

The third person flips every switch whose label is a multiple of 3 (turning some off & some on)

This continues until all 10 people pass through, the last flipping only the 10th switch.

At the end, which switches are on, and which are off?

HW: I'll move to Friday

in general, due Monday 11:33 pm

Exam: Mon Feb ~~26~~ 23

Rank and number of solutions

Observation: Given a system of linear equations, then changing it to (reduced) row echelon form doesn't change the solutions.

Rank of a system $\equiv$ # of rows in (reduced) row eschelon form.

(red. row. esh. form)

when reduce to RREF, easy to check if system is

consistent:

ex:

$$\begin{aligned} 3x &= 3 \\ 3x &= 1 \end{aligned} \rightsquigarrow \left[\begin{array}{c|c} 3 & 3 \\ 3 & 1 \end{array}\right]$$

$$-3 \left[\begin{array}{c|c} 1 & 1 \\ 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{c|c} 1 & 1 \\ 0 & -2 \end{array}\right]$$

$$\boxed{\begin{aligned} x &= 1 \\ 0 &= -2 \end{aligned}}$$

Observation:

If we have a system in (Red) REF which is inconsistent, it will have a row of form

$$[0 \text{ --- } 0 \mid a] \quad a \neq 0$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & 3 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] \longrightarrow \text{inconsistent.}$$

On the other hand, if all the bottom rows look like "$0 = 0$" then system is is always consistent.

$$\begin{array}{cccccc} x & y & z & w & & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & 3 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \text{parametrized solution}$$

$z = t \quad w = s$

$x = 3 - 2t - 5s$

$y = 7 - 3s$

$$x + 2z + 5w = 3$$
$$y + 3w = 7$$

Theorem (1.2.2)

Suppose we have a system of equations in n variables of rank r. If consistent then the solutions will involve exactly $n-r$ parameters. If $r<n$ this means we have ∞ly many solutions. If $r=n$ there is a unique solution.

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & 3 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

} rank 2

$0=0 \Rightarrow$ consistent

4 variables

$\rightsquigarrow 4-2=2$ paramtrs.

(1 paramtr for each column w/out a leading 1)

(z, w = free paramtrs)

Observation: If a system is homogeneous

$$\left[\begin{array}{c|c} \cdots & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$$

then its always consistent.

$$a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + \cdots = 0$$

$$\vdots$$

solution:

$x_1 = 0$

$x_2 = 0$

$\vdots$

$x_n = 0$

Recall: homogeneous systems are also nice because solutions form a space of vectors

Line through (2,4) & (1,3)

$$y = mx + b \xrightarrow{2,4} 4 = m(2) + b$$

$$\xrightarrow{1,3} 3 = m(1) + b$$

$$\rightarrow \begin{array}{cc} m & b \\ \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & -2 \end{array}\right]$$

$$\leftarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$m = 1$

$b = 2$

$$\boxed{y = 1x + 2}$$

Line through (1,2) & (1,4)

$$y = mx + b \xrightarrow{(1,2)} 2 = m(1) + b$$

$$\xrightarrow{(1,4)} 4 = m(1) + b$$

$$\rightarrow \begin{array}{cc} m & b \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right]$$

given two points $p=(p_1,p_2)$ $q=(q_1,q_2)$

is there a line through them of form $y=mx+b$, and if so, how many?

$$y=mx+b \xrightarrow{p} p_2 = m(p_1)+b$$

$$y=mx+b \xrightarrow{q} q_2 = m(p_2)+b$$

$$\begin{array}{ccc} m & b & \\ \end{array}$$

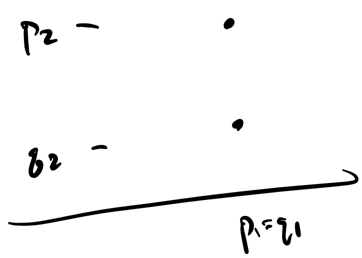
$$\left[\begin{array}{cc|c} p_1 & 1 & p_2 \\ q_1 & 1 & q_2 \end{array}\right]$$

swap variables

$$\begin{array}{cc} b & m \end{array}$$

$$\left[\begin{array}{cc|c} 1 & p_1 & p_2 \\ 1 & q_1 & q_2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & p_1 & p_2 \\ 0 & q_1-p_1 & q_2-p_2 \end{array}\right]$$



if $q_1-p_1=0$ and $q_2-p_2\neq 0$

inconsistent.

$q_1=p_1$ (same x-coord)

$p_2\neq q_2$

(diff. y-coord)

if $p_1\neq q_1$ then

$$\left[\begin{array}{cc|c} 1 & p_1 & p_2 \\ 0 & q_1-p_1 & q_2-p_2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & p_1 & p_2 \end{array}\right.$$

$$\begin{bmatrix} 0 & 1 & \Big| \frac{q_2-p_2}{q_1-p_1} \end{bmatrix}$$

there's always a unique line through any 2 points w/ <u>different</u> x-coords

of form $y=mx+b$.

or what if

$$\left[\begin{array}{cc|c} 1 & p_1 & p_2 \\ 0 & q_1-p_1 & q_2-p_2 \end{array}\right]$$

and $q_1-p_1=0$

& $q_2-p_2=0$

either $q_1=p_1$ or $q_1 \neq p_1$

$q_1-p_1=0$ $\quad$ $q_1-p_1\neq 0$

either $q_2-p_2\neq 0$ or $q_2-p_2=0$

$\Rightarrow$ contradiction (vertical line)

$$\left[\begin{array}{cc|c} 1 & p_1 & p_2 \\ 0 & 0 & 0 \end{array}\right] \quad \text{rk}=1$$

<u>only many lines through 1 point</u>

always unique soln

Start of matrix algebra

dot product

we think of coeffs of equation as a "row vector"

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 9 \end{bmatrix} \qquad \begin{matrix} x_1 + 3x_2 + 5x_3 \\ 2x_1 + 6x_2 + 9x_3 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 2 \\ 2 & 6 & 9 & 4 \end{array}\right] \qquad \begin{matrix} x_1 + 3x_2 + 5x_3 = 2 \\ 2x_1 + 6x_2 + 9x_3 = 4 \end{matrix}$$

rows $\longleftrightarrow$ coeffs   column on right = "output vector"

column vectors $\begin{bmatrix} \vdots \end{bmatrix}$ $\begin{bmatrix} \; \end{bmatrix}$

as input or output.   rows = coeffs.

$$x_1 + 3x_2 + 5x_3 = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 2 \\ 2 & 6 & 9 & 4 \end{array}\right] \qquad \begin{matrix} x_1 + 3x_2 + 5x_3 = 2 \\ 2x_1 + 6x_2 + 9x_3 = 4 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \quad \Big\} \quad \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$[2\ 6\ 9]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = T$$

Matrix: matrix mult. is a bunch of dot products.

$R_1 = (1\ 3\ 5)$

$$\begin{bmatrix} R_1 \\ \hline R_2 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} R_1 \cdot x \\ \hline R_2 \cdot x \end{bmatrix}$$

$$RX = B$$

$$\begin{bmatrix} R_1 \\ \hline R_2 \\ \hline \vdots \\ R_m \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} B \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ \hline R_2 \\ \vdots \\ \hline R_m \end{bmatrix}\left[\begin{array}{c|c|c|c} X_1 & X_2 & \cdots & X_d \end{array}\right] = \left[\begin{array}{c|c|c} R_1 \cdot X_1 & R_1 X_2 & \\ R_2 \cdot X_1 & & \cdot \\ \vdots & & \\ R_m \cdot X_1 & & \end{array}\right]$$