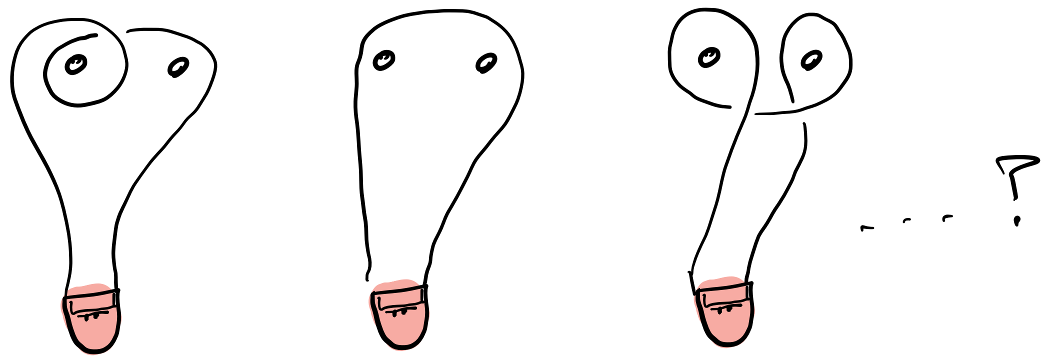

Is it possible to hang the purse from two pegs so that it stays up, but will fall if either peg is removed?

(none of the above work!)

<u>Plan:</u> Matrix algebra

Basic building block of matrix multiplication

is row × column = number

$$a \cdot x = [a_1 \; a_2 \cdots a_n] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

x_i's = variables.

$$a \cdot v = [a_1 \cdots a_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = a_1v_1 + \cdots + a_nv_n$$

v_i's = #s

Given a system:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

we "stack the notation"

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & - & - & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & - & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b_2$$

In general: if $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ $C = \begin{bmatrix} c_{11} & \cdots & c_{1r} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nr} \end{bmatrix}$

$A \cdot C$ = matrix w/ i,j entry = ith row of A $\cdot$ jth row of C

Idea: rows on left
columns on right

$$A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix} \qquad B = \begin{bmatrix} B_1 & \cdots & B_r \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} A_1 \cdot B_1 & A_1 \cdot B_2 & A_1 \cdot B_3 & \cdots & A_1 \cdot B_r \\ \vdots & & & & \vdots \\ A_m \cdot B_1 & - & & - & A_m \cdot B_r \end{bmatrix}$$

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note: for this to make sense, length of rows in A

= length of columns in B

$$\boxed{(m \times n) \cdot (n \times r) = (m \times r)}$$

<u>Note</u>: dot product has some convenient properties, that the matrix product inherits:

$$A \cdot (B_1 + B_2) = A \cdot B_1 + A \cdot B_2$$

$$(A_1 + A_2) B = A_1 B + A_2 B$$

<u>So here</u>: addition of matrices is inherited from addition of vectors

$$A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ \hline \vdots \\ \hline B_m \end{bmatrix}$$

$$A + B = \begin{bmatrix} A_1 + B_1 \\ \hline \vdots \\ \hline A_m + B_m \end{bmatrix}$$

or - slice vertically - or just add each entry.

Quick matrix theory practice

Solve $Ax = b$ inhom. system of eqns.

$(A\vec{x} = \vec{b})$ $\quad x, b$ = column vectors

Prop: if v is any soln to $Ax = b$ (i.e. $Av = b$) and w is any soln to $Ax = 0$ then $v + w$ is a soln to $Ax = b$

and conversely, every soln to $Ax = b$ has the form $x = v + w$ where w is a soln to $Ax = 0$.

Why? if $Av = b$ & $Aw = 0$

$$b = b + 0 = Av + Aw = A(v + w)$$

$v + w$ is a soln to $Ax = b$

Conversely: if v' is any other soln to $Ax = b$

then $A(v' - v) = Av' - Av = b - b = 0$

set $w = v' - v$ is a soln to $Ax = 0$

and $v' = v + w$ ✓

Remarkably matrix multiplication is associative too

$$(AB)C = A(BC)$$

Row operations

when we write $Tx = b$ T an $m \times n$ matrix

T gives a rule for taking a column w/ n entries
to a column w/ m entries.

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}$$

$$\vec{v} \mapsto T\vec{v}$$

$$\begin{array}{c} A_1 \\ \vdots \\ A_m \end{array} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} A_1 \cdot \vec{v} \\ \vdots \\ A_m \cdot \vec{v} \end{bmatrix}$$

ex: $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \longmapsto \begin{bmatrix} v_2 \\ v_1 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$ this is a function.
is it given by a matrix?

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longmapsto \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & \cdots & \\ 0 & 0 & 1 & \\ & & & 1 \\ & & & & 1 \end{bmatrix}$$

$0 + v_2 = v_2$

$v_1 + 0 = v_1$

$$\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & 0 \\ & & \ddots & & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ \hline R_2 \\ \hline \vdots \\ R_n \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \overset{T}{\longmapsto} \begin{bmatrix} \lambda v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\begin{aligned} \lambda v_1 &= \lambda v_1 + 0 v_2 + \cdots \\ v_2 &= 0 v_1 + 1 v_2 + 0 v_3 \cdots \\ v_3 &= 0 v_1 + 0 v_2 + 1 v_3 + \cdots \\ &\vdots \\ v_n &= \end{aligned}$$

$$\begin{bmatrix} \lambda & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ \hline \vdots \\ \hline R_n \end{bmatrix} = \begin{bmatrix} \lambda R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ \lambda v_1 + v_2 \end{bmatrix} = \begin{bmatrix} 1 v_1 + 0 v_2 \\ \lambda v_1 + 1 v_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & & & \\ \lambda & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ \hline R_2 \\ \hline \vdots \\ R_n \end{bmatrix} = \begin{bmatrix} R_1 \\ \hline \lambda R_1 + R_2 \\ \hline R_3 \\ \vdots \\ R_n \end{bmatrix}$$

Punchline: if $A = \begin{pmatrix} a_{11} & \cdots & \\ \vdots & & \\ \vdots & & \end{pmatrix}$ coeffs of system of linear equns

$Ax = b$

then for each elementary row operation

- swap rows (I)
- mult. row by scalar (II)
- add mult. of one row to another (III)

there's a (very simple) matrix E s.t.

$E \cdot A$ gives the transformed set of equns.

correspond "elementary matrices" of types I, II, III correspond to these.

ex:

$$2x + 3y = 5$$
$$x - y = 1$$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}_{b}$$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$E_1 =$

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \; -2 \qquad \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = E_2$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} \cdot \tfrac{1}{5} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \; 1$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix} \qquad \downarrow \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{E_4 E_3 E_2 E_1}_{E} \overset{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Av = b$$

$$r = \underbrace{EA}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} v = Eb$$

$$x_1 + 0x_2 \qquad x_1$$

$$0 + 1x_2 \qquad x_2$$

$E \rightsquigarrow$ "inverse of A"

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow$ identity.