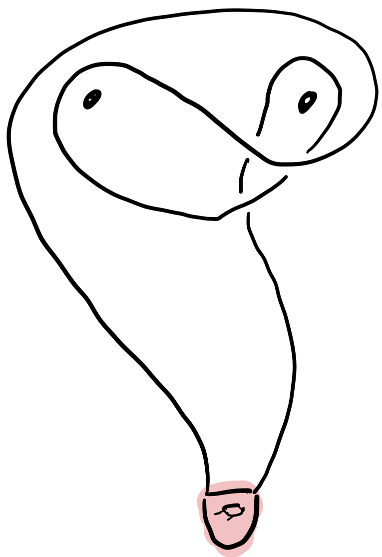
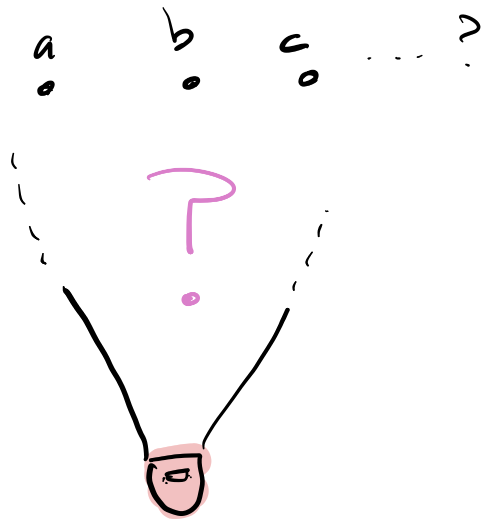




Row operations, inversion, LU factorization

elementary row op's

- swap rows I
- rescale a row II
- add mult of row to another III

corresp to "elementary matrices"

these "do" the above operation via left multiplying.

ex: $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \xrightarrow[\text{I}]{\text{swap}} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \text{J-2} \xrightarrow{\text{III}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{3-1}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{III}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$E_3E_2E_1\begin{bmatrix}2&3\\1&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Solve $\quad 2x+3y=4$

$\quad\quad\quad x+y=2 \rightsquigarrow \begin{bmatrix}2&3\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$

$$\left[\begin{array}{cc|c}2&3&4\\1&1&2\end{array}\right]\xrightarrow{E_1}\left[\begin{array}{cc|c}1&1&2\\2&3&4\end{array}\right]\xrightarrow{E_2}\left[\begin{array}{c|c}\cancel{*}&2\\&0\end{array}\right]\xrightarrow{E_3}\left[\begin{array}{cc|c}1&0&2\\0&1&0\end{array}\right]$$

$x=2$

$y=0$

$$\begin{bmatrix}2&3\\1&1\end{bmatrix}\xrightarrow[\text{I}]{\text{swap}}\begin{bmatrix}1&1\\2&3\end{bmatrix}\begin{matrix}\\-2\end{matrix}\xrightarrow{\text{III}}\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{matrix}-1\\ \\ \end{matrix}\xrightarrow{\text{III}}\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$E_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $\quad E_2=\begin{bmatrix}1&0\\-2&1\end{bmatrix}$ $\quad E_3=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$

$$\begin{bmatrix}2&3\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$$

$$\underbrace{E_3E_2E_1\begin{bmatrix}2&3\\1&1\end{bmatrix}}_{}\begin{bmatrix}x\\y\end{bmatrix}=E_3E_2E_1\begin{bmatrix}4\\2\end{bmatrix}\to E_3E_2\begin{bmatrix}2\\4\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}\leftarrow E_3\begin{bmatrix}2\\0\end{bmatrix}$$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $x=2$ $y=0$.

or: $E = E_3 E_2 E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

$E \cdot \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Identity

inverse

$E = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1}$

$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\cancel{\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Back to matrix algebra

Operations on matrices

- Can add matrices of same size (entry by entry)

  $$A+B \quad \text{if } A, B \text{ both } m\times n$$

- Scalar multiplication (can rescale all entries of a matrix)

  $$\lambda A \quad \text{if } \lambda \in \mathbb{R}$$

- Matrix multiplication $(m\times n)\cdot(n\times r) =$ an $(m\times r)$ matrix via bunch of dot products

- Matrix mult distributes over addition

- Mentioned: $A(BC)=(AB)C$

Also: Special $n\times n$ matrix $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$I_n x = y$ means
$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 \\ &\vdots \\ x_n &= y_n \end{aligned}$$

Matrices give linear transformations.

Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a function such that $T(v+w) = T(v) + T(w)$
& $T(\lambda v) = \lambda T(v)$

Recall: Matrix $\underset{m\times n}{A}$ gives a function $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ by

$$\underset{\substack{n\times 1\\ \mathbb{R}^n}}{v} \mapsto \underset{\substack{m\times 1\\ \mathbb{R}^m}}{Av} = T(v)$$

Because matrix multiplication distributes

$$T(v+w) = A\cdot(v+w) = Av + Aw = T(v) + T(w)$$

$$T(\lambda v) = A\cdot\lambda v \underset{\text{check}}{=} \lambda A\cdot v = \lambda\cdot T(v)$$

Big fact:

Matrices "are" linear transformations

Point is: given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ there's a "easy" way to get an explicit matrix

Start w/ "basis vectors" in $\mathbb{R}^n$ & $\mathbb{R}^m$

$$e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \in \mathbb{R}^n \quad e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \in \mathbb{R}^n \quad \cdots \quad e_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} \in \mathbb{R}^n$$

$$f_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \in \mathbb{R}^m \quad \cdots \quad f_m = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} \in \mathbb{R}^m$$

$v \in \mathbb{R}^n$ any vector $\quad v = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} = \begin{bmatrix}v_1\\0\\\vdots\\0\end{bmatrix} + \begin{bmatrix}0\\v_2\\0\\\vdots\end{bmatrix} + \cdots + \begin{bmatrix}0\\\vdots\\0\\v_n\end{bmatrix}$

$v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$

$$Tv = T(v_1 e_1 + \cdots + v_n e_n)$$
$$= T(v_1 e_1) + \cdots + T(v_n e_n) = v_1 T(e_1) + \cdots + v_n T(e_n)$$

if we know $T(e_i)$, know $T(\text{anything})$.

$T(e_j) \in \mathbb{R}^m$

$$T(e_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m$$

$$T(e_j) = a_{1j} f_1 + a_{2j} f_2 + a_{3j} f_3 + \cdots + a_{mj} f_m$$
$$= \underline{\sum_{i=1}^{m} a_{ij} f_i}$$

this means $T(v) = T(v_1 e_1 + \cdots + v_n e_n)$

$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$= T\left(\sum_{j=1}^{n} v_j e_j\right)$$
$$= \sum_{j=1}^{n} T(v_j e_j)$$
$$= \sum_{j=1}^{n} v_j T(e_j)$$
$$= \sum_{j=1}^{n} v_j \sum_{i=1}^{m} a_{ij} f_i$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} v_j a_{ij} f_i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} v_j a_{ij} \right) f_i$$

vector notation

$$\begin{bmatrix} \sum_{j=1}^{n} v_j a_{1j} \\ \sum_{j=1}^{n} v_j a_{2j} \\ \vdots \\ \sum_{j=1}^{n} v_j a_{mj} \end{bmatrix} = T(v)$$

$$\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

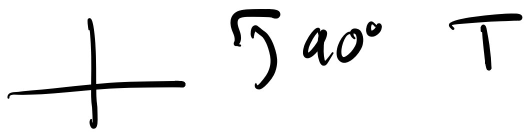
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & & & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = T(v)$$

found T given by matrix A

step back:

$$\boxed{T(e_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = j\text{th column of } A}$$

ex: rotate 90° counterclockwise in $\mathbb{R}^2$



$\rightarrow$ 1,0 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\uparrow$ 0,1

$\downarrow 90°$ $\downarrow 90°$

$\uparrow$ 0,1 $\leftarrow$ -1,0

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$