Last time:

Def A liner transformation $f: \mathbb{R}^n \to \mathbb{R}^m$ is a function satisfying the following conditions

$$\left.\begin{aligned} &- f(v+w) = f(v) + f(w) \\ &- f(\lambda v) = \lambda f(v) \end{aligned}\right\} \text{"linearity"}$$

Note: if T is an $m \times n$ matrix, it defines a liner transformation via $f(v) = Tv$ where we write v as a column vector

We showed: If f is a liner transformation, it comes from a (unique) matrix T as above.

To find the matrix T:

the columns of T are f applied to the "coordinate" basis vectors

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th position.}$$

$$3x_1 + 2x_2 = y_1$$
$$2x_1 - 4x_2 = y_2$$

$$\begin{bmatrix} 3 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

So:

- ith row of a matrix T tells you how to combine entries of input to get ith coordinate in target
- jth column of matrix T tells you where the jth basis vector goes $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ = image of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Practice:

$f(x,y) = (x, x+y) \xrightarrow{\text{matrix}}$

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ x+y \end{bmatrix}$

column $\rightarrow$ $f(1,0) = (1, 1+0) = (1,1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$f(0,1) = (0, 0+1) = (0,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

row $\searrow$

1st entry $x = 1x + 0y = [1 \;\; 0]$

2nd entry $x+y = 1x + 1y = [1 \;\; 1]$

$\left.\begin{matrix} [1 \;\; 0] \\ [1 \;\; 1] \end{matrix}\right\} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Rotate 90° clock wise

column: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1,0)$ $\quad (1,0) \xrightarrow{\text{rotate}} (0,-1) \quad \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (0,1)$ $\quad (0,1) \longmapsto (1,0) \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Elementary matrix that takes a system of 5 equations and adds the 2nd to the 4th.

= transformation that changes the 4th coord. by adding the 2nd to it.

first entry of target = first entry of source
2nd entry of target = 2nd entry of source
⋮
4th entry of target = 2nd + 4th entry of source
5th : same.

entres $x_1, x_2, \ldots, x_5$

first: $1x_1 + 0x_2 + 0x_3 + \cdots 0x_5$

second $0 \cdot - 1x_2 \quad 0$

$0 \cdots \quad 1x_3 \cdot \cdot 0$

forth $0 + 1x_2 + 0x_3 + 1x_4 + 0x_5$

$\cdot \quad - 0 \quad - \quad 1x_5$

$$\left.\right\} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ 7 \\ 1 \\ -1 \end{bmatrix}$$

entres of column

$\leadsto (3, 2, 7, 1, -1)$

coords of vector

"Multiplication is composition"

If A, B are matrices, A has n columns & B has n rows

then we can multiply AB

If B_i = ith column of B

$$B = \left[\, B_1 \,\middle|\, B_2 \,\middle|\, B_3 \,\middle|\, \cdots \,\right]$$

AB_i is a matrix · column

$$AB = \left[\, AB_1 \,\middle|\, AB_2 \,\middle|\, \cdots \,\middle|\, AB_n \,\right]$$

if $A \leadsto$ lin. trans. f
$B \leadsto$ " " g

let $C \leadsto fg$.

what's the matrix for C?

i'th column of C = fg(ith basis vector)
= f(g(ith basis vector))
= f(ith column of B) = $f(B_i)$
= AB_i
= ith column of matrix product AB !

<u>ex</u>: $A(BC) = (AB)C$

$A \leadsto f$
$B \leadsto g$
$C \leadsto h$

$f \circ (g \circ h)$ $\quad$ $(f \circ g) \circ h$

$(f \circ (g \circ h))(v) = f(g \circ h(v))$
$= f(g(h(v)))$

$((f \circ g) \circ h)v$
$= f \circ g(h(v))$
$= f(g(h(v)))$

Row operations, factorizations & inversion.

If A, B $n \times n$ matrices, we say B is the left inverse of A

if $BA = I_n$

$I_n =$ "identity matrix"

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \longleftrightarrow \text{lin. transformation } f(x) = x$$

and we write $B = A^{-1}$

<u>Question</u>: if $BA = I_n$ is it true that $AB = I_n$

in other words, if $A^{-1}A = I_n$, is $AA^{-1} = I_n$

or is $A = (A^{-1})^{-1}$

<u>ex</u>: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$

$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$

Fact: if A is an $n \times n$ matrix & B is also $n \times n$ &

$BA = I_n$ then $AB = I_n$ also

One way to find inverses & understand matrix better is via factorization/row operations

Suppose we have a matrix A, $n \times n$ & we do row ops corresponding to elementary matrices E_1, E_2, E_3, E_4 to transform A to the identity matrix

$$A \rightsquigarrow E_1 A \rightsquigarrow E_2 E_1 A \rightsquigarrow E_3 E_2 E_1 A$$

$$\Downarrow$$

$$E_4 E_3 E_2 E_1 A = I_n$$

tells you:

product $E_4 E_3 E_2 E_1 = A^{-1}$

also, inverting row ops is very easy.

$$E_4 E_3 E_2 E_1 A = I_n$$

$$\Downarrow$$

$$\cancel{E_4^{-1} E_4} E_3 E_2 E_1 A = E_4^{-1} I_n = E_4^{-1}$$

$$E_3^{-1}$$

$$E_2^{-1}$$

$$\vdots$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$A = E_1^{-1} \cdots E_4^{-1}$ "factorization"

$$Ax = b$$

$$E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} x = b$$

$$E_1^{-1} \left(E_2^{-1} E_3^{-1} E_4^{-1} x \right) = b$$

$$E_1^{-1} y = b \qquad E_2^{-1} E_3^{-1} E_4^{-1} x = a$$

$$\rightsquigarrow y = a$$