Inversion & factorization of matrices

example

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 3/2 \\ 0 & -1/2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}$$

$$E_4 E_3 E_2 E_1 A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Last time: Linear transformations ⟷ matrices
(special functions) $\mathbb{R}^n \to \mathbb{R}^m$

composition ⟷ multiplication of matrices

$$\boxed{E_4 E_3 E_2 E_1 = A^{-1}}$$

$$Ax = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{E_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{E_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{E_3} \begin{bmatrix} 0 \\ -2 \end{bmatrix} \longmapsto \begin{bmatrix} 3 \\ -2 \end{bmatrix} = x$$

$$E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \quad E_4^{-1} = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

$$E_4 E_3 E_2 E_1 A = I_2$$

$$E_3 E_2 E_1 A = E_4^{-1}$$

$$\cancel{E_3^{-1} E_3} E_2 E_1 A = E_3^{-1} E_4^{-1}$$

$$\cancel{E_2^{-1} E_2} E_1 A = E_2^{-1} E_3^{-1} E_4^{-1}$$

$$\cancel{E_1^{-1} E_1} A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

Premise:

Solving "triangular" systems is fast & robust.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

"back substitution"

$$\ldots \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 7 \\ 5 & 7 & 2 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} 1 & 3/2 \\ 0 & -1/2 \end{bmatrix} \xrightarrow{E_3} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

$$\underline{E} = \begin{bmatrix} 1/2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}}} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1/2 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & -1/2 \end{bmatrix}$$

$$A = \underset{L}{\begin{bmatrix} 2 & 0 \\ 1 & -1/2 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}}$$

"LU factorization"

<u>Algorithm</u> given $A = LU$

to solve $Ax=b$ we write $LUx=b$

let $y = Ux$

$Ly=b$

⇝ can solve (backsub.)

answer $y = a$

then solve for x by: $Ux=a$

can solve for x via forward sub.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

L U

$$Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$L\underbrace{Ux}_{} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ define $y = Ux$

$Ly = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ⇝ $y = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$

$2y_1 = 1$ ⇝ $y_1 = 1/2$

$y_1 - 1/2\, y_2 = 1$ ⇝ $y_2 = -2(1 - y_1)$

$= -2\frac{1}{2} = -1$

$$Ux = y = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + \tfrac{3}{2}x_2 &= \tfrac{1}{2} \\ x_2 &= -1 \end{aligned}\right\} \quad x_1 = \tfrac{1}{2} - \tfrac{3}{2}x_2 = \tfrac{1}{2} - \tfrac{3}{2}(-1) = \tfrac{1}{2} + \tfrac{3}{2} = 2.$$

$$x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Subspaces, rank, image, nullity

How do we specify a vector?
as an n-tuple of numbers (coords or entries)

<u>Obvious statement</u>: two vectors are the same exactly when they have the same entries.

<u>Alternate formulation</u> A vector is 0 exactly when all its entries are 0.

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \text{ if } v = w \iff v - w = 0$$

all entries of difference vector are 0

$$\begin{bmatrix} v_1 - w_1 \\ \vdots \\ v_n - w_n \end{bmatrix}$$

this relfects an important property of the "basis vectors"

$e_1, \dots, e_n$ $\quad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \dots$

Alt. formulation $\Longleftrightarrow$ if $a_1 e_1 + a_2 e_2 + \dots + a_n e_n = 0$

$\Rightarrow a_i = 0$ all i

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

<u>Def</u> A collection of vectors $v_1, \dots, v_m$ is called (linearly) independent if whenever $\sum_{i=1}^{m} a_i v_i = 0$ we have $a_i = 0$ all i.

<u>Def</u> (recall) a set V of vectors in $\mathbb{R}^n$ is called a (sub)space if the sum of any two elements of V is again in V, and any mult. of a vector in V is also in V.

$v, w \in V \Rightarrow v + w \in V$

$v \in V, \lambda \in \mathbb{R} \Rightarrow \lambda v \in V$.

<u>Def</u> if V is a subspace of $\mathbb{R}^n$, we say that $v_1, \dots, v_m$ spans V

if every v in V can be written in form

$v = a_1 v_1 + \cdots + a_m v_m$

<u>Def</u> A basis for a subspace V is a spanning, independent set

<u>Fact:</u> these always exist & all have the same size = "dimension of V"

a basis = spanning & independent = maximal independent

= minimal spanning set

<u>Main point:</u> if T is a linear transformation with matrix A

<u>Def</u>: $im(T)$ image of T = $\{T(x) \mid x \text{ a vector}\}$

= range of T

$null(T)$ nullspace of T = $\{x \mid T(x) = 0\}$

= solns to $Ax = 0$.

parametric solns

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = 0$$

$x_5 = 0$

$x_4 = s$

$x_3 = -x_4 = -s$

$x_2 = t$

$x_1 = -2x_2 - 4x_4 = -2t - s$

$$x = \begin{bmatrix} -2t - s \\ t \\ -s \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s$$