This week

office hours Thurs 9am → DRL 4E1A

Friday 10am } or by appt.

Review this wednesday

review sheet by tomorrow

Cheat sheet (1 page letter or A4)

front & back

no <u>electronics</u>, cellphones in bag under seat etc.

bring pens, pencils, etc.

<u>Last time</u>:

<u>Def</u> A set of vectors $v_1, \ldots, v_n$ is called <u>independent</u> if whenever $a_1v_1 + \cdots + a_nv_n = 0$ we have $a_1 = a_2 = \cdots = a_n = 0$.

ex: $v_i = e_i$ $\quad a_1v_1 + \cdots + a_nv_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$= \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$

<u>Def</u> A set of vectors $v_1, \ldots, v_n$ spans a subspace V (in $\mathbb{R}^n$) if every vector in V can be written as $a_1v_1 + \cdots + a_nv_n$

<u>Def</u> A basis for a subspace $V \subseteq \mathbb{R}^n$ is an independent, spanning set of vectors in V.

Remark Basis = max'l independent set
= min'l spanning set.

If we have a system of homogeneous eqns $Ax=0$ then the solutions form a subspace.

How to think about this?

$A \longleftrightarrow T: \mathbb{R}^n \to \mathbb{R}^m$

m×n matrix

$Ax=0 \longleftrightarrow T(x)=0$ def of lin trans.

$T(x+x') = T(x)+T(x')$
$= 0+0 = 0$
if $T(x)=0=T(x')$

if $Ax=0$ & $Ax'=0$
then $A(x+x')$
$= Ax + Ax' = 0+0=0.$

$A(\lambda x) = \lambda(Ax) = \lambda 0$
$=0.$

Vocabulary:

Solns to $Ax=0$ = null space of A
= kernel of T (or A)

ex: (parametric solns ⇝ basis for nullspace)

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix}$$

$x_5 = 0$

$x_3 + x_4 = 0 \to x_3 = -x_4$

$x_1 + 2x_2 + 4x_4 = 0 \quad x_1 = -2x_2 - 4x_4$

$x_4 = s$
$x_2 = t$

$\underbrace{\quad}_{\text{determined part}}$ $\underbrace{\qquad\qquad}_{\text{free parameter part}}$

$x_1 = -2t - 4s$
$x_2 = t$
$x_3 = -s$
$x_4 = s$
$x_5 = 0$

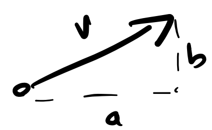
$$\rightsquigarrow x = \begin{bmatrix} -2t-4s \\ t \\ -s \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4s \\ 0 \\ -s \\ s \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s$$

Remember: $V \subseteq \mathbb{R}^n$ is a *subspace* if whenever $v, w \in V$ and $\lambda \in \mathbb{R}$ then $v + w \in V$ & $\lambda v \in V$.

Vectors & Geometry

$\mathbb{R}^2$



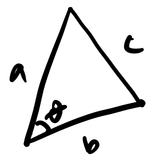
$v = (a, b)$ $\quad \|v\| = \text{length of } v$
$= \sqrt{a^2 + b^2}$

length: $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

$$\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = [x_1 \cdots x_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x \cdot x$$

angles:

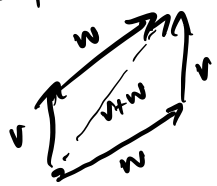
recall



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

sidebar:

vector addition

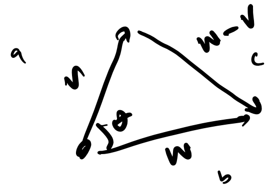


parallelogram law

$v + w = w + v$

-subtraction law (end to end)

$w - v$ = thing you add to v to get w.



law of cosines:

$$\|w - v\|^2 = \|v\|^2 + \|w\|^2 - 2\,\|v\|\,\|w\|\cos\theta$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$\|v\|^2 = v \cdot v$

$$(w - v)\cdot(w - v) = v\cdot v + w\cdot w - 2\|v\|\,\|w\|\cos\theta$$

$$= w\cdot(w - v) - v\cdot(w - v)$$

$$= w\cdot w - v\cdot w - v\cdot w + v\cdot v$$

$$\cancel{v\cdot v} + \cancel{w\cdot w} - \cancel{2}\, v\cdot w = \cancel{v\cdot v} + \cancel{w\cdot w} - \cancel{2}\,\|v\|\,\|w\|\cos\theta$$

$$\boxed{v\cdot w = \|v\|\,\|w\|\cos\theta}$$

$$\frac{v \cdot w}{\|v\| \, \|w\|} = \cos\theta$$

$$\cos\theta = 0 \iff v \perp w \iff v \cdot w = 0$$

given a single eqn $\quad v_1x_1 + v_2x_2 + \cdots + v_nx_n = 0$

$v_1, v_2, \ldots, v_n \in \mathbb{R}$

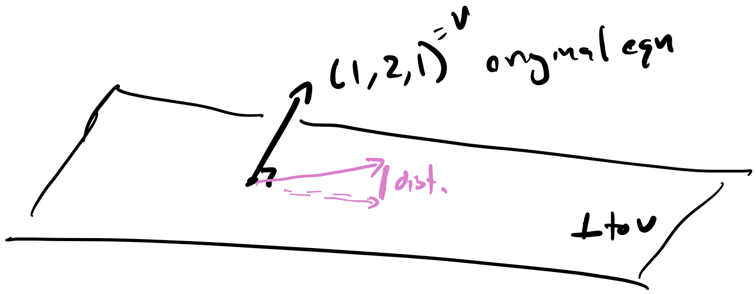
$$\Updownarrow$$

$$v \cdot x = 0$$

$$\Updownarrow$$

$$v \perp x$$

$$x_1 + 2x_2 + x_3 = 0 \qquad x = (-1, 1, 0)$$



~~**Fact:** given an eqn $v \cdot x = 0$~~

~~v = coeffs $\quad x$ = vars.~~

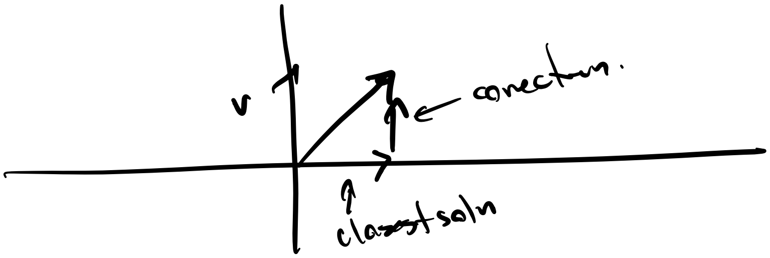
Given any two vectors v & x
can write $x = x^{\parallel} + x^{\perp}$ where $x^{\parallel}$ is parallel to v
& $x^{\perp}$ is perp. to v

$x^{\parallel}$ = correction needed for soln

$x^{\perp}$ = closest soln.

$\mathbb{R}^2$ eqn is $y = 0$ $v = (0,1)$

$$v \cdot (x,y) = y = 0$$



$$x = x^{\parallel} + x^{\perp}$$

$x^{\parallel}$ parallel to v

$(x^{\parallel} = \lambda v)$

$x^{\perp}$ perp to v.

$$v \cdot x = (\lambda v + x^{\perp}) \cdot v$$

$$v \cdot x = \lambda v \cdot v + 0$$

$$\lambda = \frac{v \cdot x}{v \cdot v} = \frac{v \cdot x}{\|v\|^2}$$

$$\boxed{\begin{aligned} x^{\parallel} &= \lambda v = \frac{v \cdot x}{v \cdot v} v \\ x^{\perp} &= x - x^{\parallel} = x - \frac{v \cdot x}{v \cdot v} v \end{aligned}}$$