Puzzles:

Given any number n, can you always find a number of the form $\underbrace{1111\cdots11}_{\text{some 1's}}\underbrace{00\cdots0}_{\text{some 0's}}$ which is a multiple of n?

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ex $n=2$ 10

$n=3$ 111 ✓ $= 3\times 37$

$n=4$ 100 ✓

$n=5$ 10

⋮

Short assignment Friday 11:33pm

extra office hour Thus TBA

Geometry of $\mathbb{R}^n$

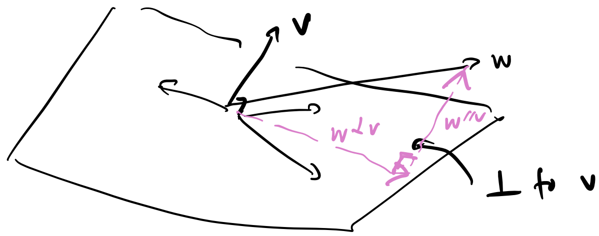
Recall: if $v, w \in \mathbb{R}^n$ and $v \neq 0$ we say w is parallel to v if $w = \lambda v$ some $\mathbb{R}$

We say v & w are perp. if $v \cdot w = 0$

Showed: $(v \neq 0)$ we can always write

$$w = w^{\perp} + w^{\parallel}$$
$$= w^{\perp v} + w^{\parallel v}$$

where $w^{\perp v}$ is perp to v & $w^{\parallel v}$ is $\parallel$ to v.



$$w^{\parallel v} = \left(\frac{w\cdot v}{v\cdot v}\right) v = \frac{w\cdot v}{\|v\|^2}\, v$$

scalar

$$= \frac{w\cdot v}{\|v\|}\cdot\frac{v}{\|v\|}$$

$$w^{\perp v} = w - w^{\parallel v}$$
$$= w - \left(\frac{w\cdot v}{v\cdot v}\right) v$$

Def a set of vectors $v_1, \dots, v_m$
is called (mutually or pairwise) orthogonal if

$v_i \cdot v_j = 0$ when $i \neq j$

is called orthonormal if also $\|v_i\| = 1$.

(properties of the standard basis vectors e_i in $\mathbb{R}^n$)

Lemma: if $v_1, \dots, v_m$ are nonzero, orthogonal vectors
then they are independent.

Proof: we'll show: if $a_1v_1 + \dots + a_mv_m = 0$ then each $a_i = 0$.

but: dot each side w/ v_i

$$v_i(a_1v_1 + \dots + a_mv_m) = v_i \cdot 0$$
$$a_1\cancel{(v_i\cdot v_1)} + \dots\ a_i(v_i \cdot v_i) + \dots + a_m\cancel{(v_i\cdot v_m)} = 0$$
$$a_i\|v_i\|^2 = 0 \Rightarrow a_i = 0$$

(each i).

done!

If $W \subseteq \mathbb{R}^n$ is some subspace spanned by $w_1, \ldots, w_m$ then we can find a set of vectors $u_1, \ldots, u_r \in W$, $r \leq m$ which span & are orthogonal (or orthonormal)
(& hence independent)

<u>Proof</u> (Graham-Schmidt)

Iterative process to change w's to u's:

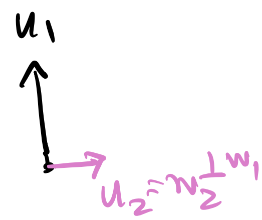
$u_1 = w_1$

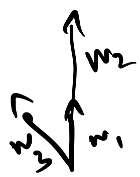
$u_2 = w_2^{\perp u_1}$ ← if w_2 not parallel to w_1, else $u_2 = w_3^{\perp u_1}$

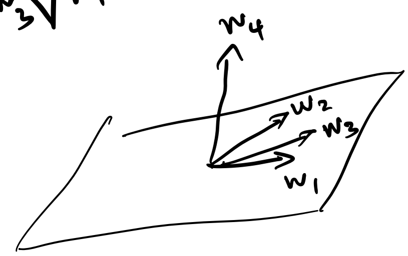
$u_3 = \left(w_3^{\perp u_1}\right)^{\perp u_2}$

why does this work?

$u_4 = \cdots$







why does this work?

$u_1 = w_1$

$u_2 = w_2^{\perp u_1} = w_2 - \left(\frac{w_2 \cdot u_1}{u_1 \cdot u_1} u_1\right) = w_2 - \left(\frac{w_2 \cdot w_1}{w_1 \cdot w_1}\right) w_1$

$u_i = \left(\left(\left(w_i^{\perp u_1}\right)^{\perp u_2}\right)^{\perp u_3} \cdots\right)^{\perp u_{i-1}}$ either 0 so skip or add on.

$$= w_i - \lambda_1 u_1 - \lambda_2 u_2 \cdots - \lambda_{i-1} u_{i-1}$$

$$= w_i - \lambda_1 w_1 - \lambda_2 (w_2 - ? w_1) - \lambda_3 (w_3 - \lambda_1 w_2 - \lambda w_1)$$

$$= w_i - \text{comb of } w_1 \cdots w_{i-1}$$

Claim: can always write u_i's in terms of w_i's &, vise versa.

So same span.

Worksheet 5

prob 1 found a couple explicit vectors

w_1, w_2 spanned same as v_1, v_2

$w_1 = (1,4,0)$ $\quad w_2 = (-4/17, 1/17, 1)$

prob 2 find w_3 perp to both w_1, w_2

~~$(w_1 \times w_2 = w_3)$~~

$$\begin{bmatrix} 1 & 4 & 0 \\ -4/17 & 1/17 & 1 \end{bmatrix} \begin{bmatrix} w_3 \end{bmatrix} = 0 \qquad (1,0,0) = v_3$$

Side comment: if have a system of eqns

$$\begin{bmatrix} R_1 \\ \hline R_2 \\ \hline \vdots \\ R_m \end{bmatrix} x = 0$$

where vectors representing R_i's are orthogonal

then can get solns by taking arb. vector x_0

& gram-schmidt.

$$\left(\cdots\left(x_0^{\perp R_1}\right)^{\perp R_2}\cdots\right)^{R_m} = x$$

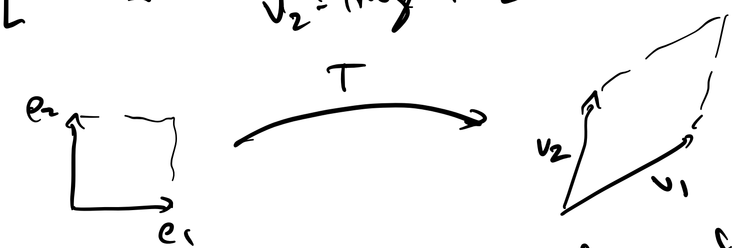
Side topic: Determinants

Geometrically: determinant of an $n \times n$ matrix

$$\det\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = $$ "signed volume" ~~volume~~ of "parallelepiped" from v_i's

can think of this as the ratio of volume of where the unit cube goes

$T \leftrightarrow \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ $v_1 = $ image of e_1, $v_2 = $ image of e_2



$$\det = \frac{\text{area of image of shape}}{\text{area of shape}} \ (\pm 1)$$

if A matrix $\longleftrightarrow$ T lin. transf.

$$\det(A) = \frac{\mathrm{area}(T(\text{shape}))}{\mathrm{are}(\text{shape})}$$

$$\det T = \frac{\mathrm{vol}(\ \ )}{\mathrm{vol}(\ \ )}$$

$$\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

det (swap coords) = -1

det (rescale an axis by λ) = λ

det of shear = 1

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$