

MATH 3120, Linear Algebra, Spring 2026, Exam 3

Review Sheet Solution Guide

These are solution sketches. In many problems, there is more than one correct way to do the computation.

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \end{bmatrix}$.

(a) Put A into reduced row echelon form.

Solution. Subtract row 1 from row 3. The new row 3 equals row 2, so we get one zero row after subtracting row 2 from row 3:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then replace row 1 by row 1 minus 2 times row 2:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Exhibit a basis for the image of A .

Solution. The rref has two nonzero rows, so the rank is 2. The first two columns of A are independent (neither is a scalar multiple of the other), so they span a 2-dimensional space. Since the rank is 2, any two independent columns span the entire image. Therefore a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

(c) Exhibit a basis for the kernel of A .

Solution. From the rref, the free variables are x_3 and x_4 , satisfying

$$x_1 - 2x_3 - x_4 = 0, \quad x_2 + x_3 + x_4 = 0.$$

Let $x_3 = s$ and $x_4 = t$. Then

$$x = (2s + t, -s - t, s, t) = s(2, -1, 1, 0) + t(1, -1, 0, 1).$$

So a basis for the kernel is

$$\{(2, -1, 1, 0), (1, -1, 0, 1)\}.$$

(d) What are the rank and nullity of A ?

Solution. From the above, the rank is 2 (two nonzero rows in the rref) and the nullity is 2 (two free variables).

2. Suppose B is obtained from a matrix A by doing row operations, and C is obtained from A by doing column operations.

(a) Compare A and B . Which space is the same for these two matrices: the image, the kernel, or both? Explain.

Solution. A and B have the same *kernel*. Row operations correspond to multiplying on the left by an invertible matrix E , so $B = EA$. This is a change of basis on the output side: it does not affect what inputs give zero output, so the kernel is unchanged. The images can differ as subspaces (though they have the same dimension), since E transforms the output.

- (b) Compare A and C . Which space is the same for these two matrices: the image, the kernel, or both? Explain.

Solution. A and C have the same *image*. Column operations correspond to multiplying on the right by an invertible matrix F , so $C = AF$. This is a change of basis in the domain: the columns of C are just an invertible recombination of the columns of A , so their span is the same. The kernels can differ (though they have the same dimension).

3. Let $u_1 = (1, 1, 0)$, $u_2 = (1, -1, 2)$, and let U be the span of u_1 and u_2 .

- (a) Check that u_1 and u_2 are perpendicular.

Solution.

$$u_1 \cdot u_2 = 1 \cdot 1 + 1 \cdot (-1) + 0 \cdot 2 = 0.$$

So the vectors are perpendicular.

- (b) Find the vector in U which is as close as possible to $v = (3, 0, 1)$.

Solution. Since u_1 and u_2 are already perpendicular, the closest vector in their span to v is the sum of the two projections:

$$\frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2.$$

We have

$$v \cdot u_1 = 3, \quad u_1 \cdot u_1 = 2, \quad v \cdot u_2 = 5, \quad u_2 \cdot u_2 = 6.$$

Therefore the closest vector is

$$\frac{3}{2}(1, 1, 0) + \frac{5}{6}(1, -1, 2) = \left(\frac{7}{3}, \frac{2}{3}, \frac{5}{3}\right).$$

- (c) Find the distance from v to U .

Solution. The leftover (perpendicular) component is

$$v - \left(\frac{7}{3}, \frac{2}{3}, \frac{5}{3}\right) = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right).$$

So the distance is

$$\left\| \left(\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right) \right\| = \sqrt{3 \cdot \frac{4}{9}} = \frac{2\sqrt{3}}{3}.$$

4. Let

$$U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}.$$

Find an orthonormal basis for U . You may choose any convenient vectors in U ; do not feel tied to a particular algorithm.

Solution. We just need two perpendicular vectors in U . Here are two simple choices picked by inspection:

$$w_1 = (1, -1, 0), \quad w_2 = (1, 1, -2).$$

Both are in U : $1 + (-1) + 0 = 0$ and $1 + 1 + (-2) = 0$. They are perpendicular:

$$w_1 \cdot w_2 = 1 - 1 + 0 = 0.$$

Their lengths are $\|w_1\| = \sqrt{2}$ and $\|w_2\| = \sqrt{6}$, so one orthonormal basis for U is

$$\left\{ \frac{1}{\sqrt{2}}(1, -1, 0), \quad \frac{1}{\sqrt{6}}(1, 1, -2) \right\}.$$

5. Let P_2 be the vector space of polynomials of degree at most 2. Define

$$T : P_2 \rightarrow P_2, \quad T(f) = f'(x) + f(0)(1+x).$$

Use the basis $B = (1, x, x^2)$ for both the input and output.

(a) Find the matrix ${}_B T_B$.

Solution. Apply T to each basis vector:

$$T(1) = 0 + (1+x) = 1+x, \quad T(x) = 1 + 0 \cdot (1+x) = 1, \quad T(x^2) = 2x + 0 \cdot (1+x) = 2x.$$

Writing these in the basis $(1, x, x^2)$ gives coordinate columns $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, so

$${}_B T_B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) What are the rank and nullity of T ?

Solution. The first two columns of ${}_B T_B$ are independent, and the third row is all zeros, so the rank is 2. Since P_2 has dimension 3, the nullity is 1.

(c) Is T invertible?

Solution. No. The nullity is 1, so T has a nontrivial kernel. (Concretely, write $f = a + bx + cx^2$ and solve $T(f) = 0$: this gives $a + b = 0$ and $a + 2c = 0$, so the kernel is spanned by $2 - 2x - x^2$.)

6. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Let $B = (b_1, b_2)$ where $b_1 = (1, 1)$, $b_2 = (1, -1)$.

(a) Find ${}_E T_B$, where E is the standard basis of $\mathbb{R}^2$.

Solution. Apply T to the basis vectors of B :

$$T(1, 1) = (3, 3), \quad T(1, -1) = (1, -1).$$

Writing these as standard coordinate columns,

$${}_E T_B = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}.$$

(b) Find ${}_B T_B$.

Solution. Now express the outputs in the basis B :

$$(3, 3) = 3(1, 1) + 0(1, -1), \quad (1, -1) = 0(1, 1) + 1(1, -1).$$

Therefore

$${}_B T_B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

7. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by the standard matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Let the input basis be $B = ((1, 1), (1, 0))$ and let the output basis be $C = ((1, 0), (1, 1))$. Find the matrix ${}_CT_B$. You may use the change of basis formula if you want.

Solution. Let P_B and P_C be the matrices whose columns are the vectors in B and C :

$$P_B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad P_C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The change of basis formula gives ${}_CT_B = P_C^{-1}AP_B$. Now

$$AP_B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}, \quad P_C^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Thus

$${}_CT_B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

8. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find a basis B for $\mathbb{R}^3$ and a basis C for $\mathbb{R}^2$ so that the matrix ${}_CT_B$ has the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution. We want two basis vectors in the domain that map to the two standard basis vectors of $\mathbb{R}^2$, and a third that maps to zero. Notice that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For a kernel vector, solve $x_1 + 2x_2 = 0$, $x_2 + x_3 = 0$. Taking $x_2 = 1$ gives $(-2, 1, -1)$. So take

$$B = ((1, 0, 0), (0, 0, 1), (-2, 1, -1))$$

and C the standard basis of $\mathbb{R}^2$. Then ${}_CT_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

9. Consider the matrix $A = \begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix}$.

(a) Is 1 an eigenvalue of A ?

Solution. Check whether $A - I$ is singular:

$$\det(A - I) = \det \begin{bmatrix} 3 & 4 \\ 7 & 15 \end{bmatrix} = 45 - 28 = 17 \neq 0.$$

So 1 is *not* an eigenvalue.

(b) Is 2 an eigenvalue of A ?

Solution. Check whether $A - 2I$ is singular:

$$\det(A - 2I) = \det \begin{bmatrix} 2 & 4 \\ 7 & 14 \end{bmatrix} = 28 - 28 = 0.$$

So 2 *is* an eigenvalue.

- (c) Find all eigenvalues of A .

Solution. The characteristic polynomial is

$$\det(A - \lambda I) = (4 - \lambda)(16 - \lambda) - 28 = \lambda^2 - 20\lambda + 36 = (\lambda - 2)(\lambda - 18).$$

The eigenvalues are 2 and 18.

10. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

- (a) Is A diagonalizable? Why or why not?

Solution. The characteristic polynomial is

$$\det(A - \lambda I) = (1 - \lambda)(4 - \lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

The eigenvalues are 2 and 3. Since a 2×2 matrix with two distinct eigenvalues is always diagonalizable, yes, A is diagonalizable.

- (b) Exhibit a matrix P such that $P^{-1}AP$ is diagonal.

Solution. For $\lambda = 2$: $A - 2I = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$, so an eigenvector is $(2, 1)$.

For $\lambda = 3$: $A - 3I = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$, so an eigenvector is $(1, 1)$.

Take

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{so that} \quad P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

11. Suppose A is a matrix such that $P^{-1}AP$ is diagonal for some orthogonal matrix P . Is it true that A is symmetric? Explain why or why not.

Solution. Yes, A must be symmetric. Since P is orthogonal, $P^{-1} = P^T$. Writing $D = P^{-1}AP$, we get $A = PDP^T$. A diagonal matrix satisfies $D^T = D$, so

$$A^T = (PDP^T)^T = PD^T P^T = PDP^T = A.$$

So A is symmetric.

12. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}.$$

Use diagonalization to find a formula for $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for positive integers k . Do not multiply A by itself k times.

Hints: eigenvectors for this matrix include $(2, 1)$ and $(1, 1)$. First write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a combination of those two eigenvectors. Then think about what A^k does to each eigenvector.

Solution. From the diagonalization in the previous problem,

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad A = PDP^{-1}.$$

Since $\det(P) = 1$, we have $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, so

$$P^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then

$$A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = PD^k P^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = P \begin{bmatrix} 2^k \\ -3^k \end{bmatrix} = \begin{bmatrix} 2^{k+1} - 3^k \\ 2^k - 3^k \end{bmatrix}.$$

13. Let

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(a) Check that the columns of P are orthonormal.

Solution. The columns are $\frac{1}{\sqrt{2}}(1, 1)$ and $\frac{1}{\sqrt{2}}(1, -1)$. Each has length 1, and their dot product is $\frac{1}{2}(1 - 1) = 0$. So the columns are orthonormal.

(b) Check that the rows of P are orthonormal.

Solution. The rows are $\frac{1}{\sqrt{2}}(1, 1)$ and $\frac{1}{\sqrt{2}}(1, -1)$, the same two vectors as the columns. So the rows are also orthonormal.

(c) What is P^{-1} ?

Solution. Since P is orthogonal, $P^{-1} = P^T$. In this example $P^T = P$ (the matrix equals its own transpose), so $P^{-1} = P$.

14. Orthogonally diagonalize the symmetric matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

In other words, find an orthogonal matrix Q so that $Q^{-1}AQ$ is diagonal.

Solution. Check that $(1, 1)$ and $(1, -1)$ are eigenvectors:

$$A(1, 1) = (3, 3) = 3(1, 1), \quad A(1, -1) = (1, -1) = 1 \cdot (1, -1).$$

After normalizing, set

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then Q is orthogonal and $Q^{-1}AQ = Q^T AQ = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

15. Consider the symmetric matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$.

(a) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A .

Solution. The first coordinate decouples from the last two, so $(1, 0, 0)$ is an eigenvector with eigenvalue 2.

For the bottom 2×2 block $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, the eigenvectors are $(1, 1)$ with eigenvalue 4 and $(1, -1)$ with eigenvalue -2 . Lifting to $\mathbb{R}^3$ gives $(0, 1, 1)$ and $(0, 1, -1)$.

After normalizing all three, an orthonormal eigenbasis is

$$\left\{ (1, 0, 0), \frac{1}{\sqrt{2}}(0, 1, 1), \frac{1}{\sqrt{2}}(0, 1, -1) \right\}.$$

(b) Find an orthogonal matrix Q so that $Q^{-1}AQ$ is diagonal.

Solution. Use the orthonormal eigenvectors as columns:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then $Q^{-1}AQ = Q^T A Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

16. Consider the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(a) Starting with $v = (1, 0)$, compute the first two directions you get by repeating the rule

$$v \mapsto \frac{Av}{\|Av\|}.$$

Solution. Starting from $(1, 0)$: $A(1, 0) = (1, 3)$, so the first new direction is $\frac{(1, 3)}{\sqrt{10}}$.

For the second step, scale does not matter, so apply A to $(1, 3)$: $A(1, 3) = (10, 6)$, giving direction $\frac{(10, 6)}{\sqrt{136}} = \frac{(5, 3)}{\sqrt{34}}$.

(b) What eigenvector should this process move toward?

Solution. The eigenvectors of A are $(1, 1)$ with eigenvalue 4 and $(1, -1)$ with eigenvalue -2 . Since $|4| > |-2|$, power iteration moves toward the direction of $(1, 1)$.

(c) What happens if you start with $v = (1, -1)$ instead?

Solution. We have $A(1, -1) = (-2, 2) = -2(1, -1)$, so $(1, -1)$ is already an eigenvector. The normalized vector stays on the same line, possibly switching sign at each step.

(d) What would you expect to happen if you chose a starting vector at random?

Solution. A random vector will almost surely have a nonzero component in the $(1, 1)$ direction. Since $|4| > |-2|$, repeated multiplication amplifies that component relative to the $(1, -1)$ component, so the directions will almost surely converge toward the line spanned by $(1, 1)$.

17. Suppose A is a symmetric matrix, and suppose v_0 is an eigenvector for A with eigenvalue λ . Let $W = \{w \in \mathbb{R}^n : v_0 \cdot w = 0\}$.

Explain why, if $w \in W$, then $Aw \in W$.

Solution. Take any $w \in W$; we need $Aw \perp v_0$. Compute:

$$v_0 \cdot Aw = Av_0 \cdot w = (\lambda v_0) \cdot w = \lambda(v_0 \cdot w) = 0,$$

where the first equality uses symmetry of A (so $v_0 \cdot Aw = Av_0 \cdot w$) and the second uses the eigenvector condition. So $Aw \in W$.

18. Find a singular value decomposition for $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$.

Use the method from class: start by finding eigenvectors of $A^T A$.

Solution. Compute

$$A^T A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}.$$

Eigenvalues: 25 and 9, with eigenvectors $(1, 1)$ and $(1, -1)$. Normalizing:

$$v_1 = \frac{1}{\sqrt{2}}(1, 1), \quad v_2 = \frac{1}{\sqrt{2}}(1, -1), \quad \sigma_1 = 5, \quad \sigma_2 = 3.$$

Then

$$Av_1 = \frac{1}{\sqrt{2}}(5, 5, 0), \quad u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{2}}(1, 1, 0),$$

$$Av_2 = \frac{1}{\sqrt{2}}(1, -1, 4), \quad u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{3\sqrt{2}}(1, -1, 4).$$

We need a third unit vector $u_3 \in \mathbb{R}^3$ perpendicular to u_1 and u_2 ; one choice is $u_3 = \frac{1}{3}(2, -2, -1)$.

So one SVD is $A = Q\Sigma P^T$ with

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & -\frac{1}{3} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}, \quad P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

19. Let A be an $m \times n$ matrix. Suppose $v_1, \dots, v_n$ is an orthonormal basis of eigenvectors for $A^T A$, and $A^T A v_i = \lambda_i v_i$.

(a) Explain why each $\lambda_i \geq 0$.

Solution. Since $\|v_i\| = 1$,

$$\lambda_i = \lambda_i(v_i \cdot v_i) = v_i \cdot (\lambda_i v_i) = v_i \cdot (A^T A v_i) = (A v_i) \cdot (A v_i) = \|A v_i\|^2 \geq 0.$$

(b) Explain why the vectors $A v_i$ are perpendicular to each other, except for the ones that are zero.

Solution. For $i \neq j$,

$$A v_i \cdot A v_j = v_i \cdot (A^T A v_j) = v_i \cdot (\lambda_j v_j) = \lambda_j(v_i \cdot v_j) = 0,$$

since $v_1, \dots, v_n$ are orthonormal. So the nonzero vectors among the $A v_i$ are perpendicular to each other.

20. Suppose $A = Q \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T$,

where P and Q are orthogonal matrices. Write the columns of P as v_1, v_2, v_3 , and write the columns of Q as u_1, u_2, u_3 .

(a) What are $A v_1, A v_2, A v_3$, in terms of the vectors u_1, u_2, u_3 ?

Solution. Since P is orthogonal, $P^T = P^{-1}$. The columns of P satisfy $P e_i = v_i$, so $P^{-1} v_i = P^T v_i = e_i$. Therefore

$$A v_i = Q \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T v_i = Q \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} e_i.$$

Reading off the columns of Σ :

$$A v_1 = 5 u_1, \quad A v_2 = 3 u_2, \quad A v_3 = 0.$$

(b) What are the rank, image, and kernel of A ?

Solution. There are two nonzero singular values, so the rank is 2. The image is $\text{span}\{u_1, u_2\}$ and the kernel is $\text{span}\{v_3\}$.

(c) Find $A(2v_1 - v_2 + 4v_3)$ and its length.

Solution. By linearity,

$$A(2v_1 - v_2 + 4v_3) = 2A v_1 - A v_2 + 4A v_3 = 10u_1 - 3u_2.$$

Since u_1 and u_2 are perpendicular unit vectors, the length is $\sqrt{10^2 + 3^2} = \sqrt{109}$.