

MATH 3120, Linear Algebra, Spring 2026, Exam 3

Review Sheet

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \end{bmatrix}$.

(a) Put A into reduced row echelon form.

(b) Exhibit a basis for the image of A .

(c) Exhibit a basis for the kernel of A .

(d) What are the rank and nullity of A ?

2. Suppose B is obtained from a matrix A by doing row operations, and C is obtained from A by doing column operations.

(a) Compare A and B . Which space is the same for these two matrices: the image, the kernel, or both? Explain.

(b) Compare A and C . Which space is the same for these two matrices: the image, the kernel, or both? Explain.

3. Let $u_1 = (1, 1, 0)$, $u_2 = (1, -1, 2)$, and let U be the span of u_1 and u_2 .

(a) Check that u_1 and u_2 are perpendicular.

(b) Find the vector in U which is as close as possible to $v = (3, 0, 1)$.

(c) Find the distance from v to U .

4. Let $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.

Find an orthonormal basis for U . You may choose any convenient vectors in U ; do not feel tied to a particular algorithm.

5. Let P_2 be the vector space of polynomials of degree at most 2. Define

$$T : P_2 \rightarrow P_2, \quad T(f) = f'(x) + f(0)(1 + x).$$

Use the basis $B = (1, x, x^2)$ for both the input and output.

- (a) Find the matrix ${}_B T_B$.

- (b) What are the rank and nullity of T ?

- (c) Is T invertible?

6. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by the standard matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Let $B = (b_1, b_2)$ where $b_1 = (1, 1)$, $b_2 = (1, -1)$.

(a) Find ${}_E T_B$, where E is the standard basis of $\mathbb{R}^2$.

(b) Find ${}_B T_B$.

7. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Let the input basis be $B = ((1, 1), (1, 0))$ and let the output basis be $C = ((1, 0), (1, 1))$.

Find the matrix ${}_C T_B$. You may use the change of basis formula if you want.

8. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Find a basis B for $\mathbb{R}^3$ and a basis C for $\mathbb{R}^2$ so that the matrix ${}_C T_B$ has the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

9. Consider the matrix $A = \begin{bmatrix} 4 & 4 \\ 7 & 16 \end{bmatrix}$.

(a) Is 1 an eigenvalue of A ?

(b) Is 2 an eigenvalue of A ?

(c) Find all eigenvalues of A .

10. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(a) Is A diagonalizable? Why or why not?

(b) Exhibit a matrix P such that $P^{-1}AP$ is diagonal.

11. Suppose A is a matrix such that $P^{-1}AP$ is diagonal for some orthogonal matrix P . Is it true that A is symmetric? Explain why or why not.

12. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

Use diagonalization to find a formula for $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for positive integers k . Do not multiply A by itself k times.

Hints: eigenvectors for this matrix include $(2, 1)$ and $(1, 1)$. First write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a combination of those two eigenvectors. Then think about what A^k does to each eigenvector.

13. Let $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(a) Check that the columns of P are orthonormal.

(b) Check that the rows of P are orthonormal.

(c) What is P^{-1} ?

14. Orthogonally diagonalize the symmetric matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

In other words, find an orthogonal matrix Q so that $Q^{-1}AQ$ is diagonal.

15. Suppose A is a symmetric matrix, and suppose v_0 is an eigenvector for A with eigenvalue λ . Let $W = \{w \in \mathbb{R}^n : v_0 \cdot w = 0\}$.

Explain why, if $w \in W$, then $Aw \in W$.

16. Consider the symmetric matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$.

(a) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A .

(b) Find an orthogonal matrix Q so that $Q^{-1}AQ$ is diagonal.

17. Consider the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(a) Starting with $v = (1, 0)$, compute the first two directions you get by repeating the rule

$$v \mapsto \frac{Av}{\|Av\|}.$$

(b) What eigenvector should this process move toward?

(c) What happens if you start with $v = (1, -1)$ instead?

(d) What would you expect to happen if you chose a starting vector at random?

18. Find a singular value decomposition for $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$.

Use the method from class: start by finding eigenvectors of $A^T A$.

19. Let A be an $m \times n$ matrix. Suppose $v_1, \dots, v_n$ is an orthonormal basis of eigenvectors for $A^T A$, and $A^T A v_i = \lambda_i v_i$.

(a) Explain why each $\lambda_i \geq 0$.

(b) Explain why the vectors Av_i are perpendicular to each other, except for the ones that are zero.

20. Suppose $A = Q \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T$,

where P and Q are orthogonal matrices. Write the columns of P as v_1, v_2, v_3 , and write the columns of Q as u_1, u_2, u_3 .

(a) What are Av_1, Av_2, Av_3 , in terms of the vectors u_1, u_2, u_3 ?

(b) What are the rank, image, and kernel of A ?

(c) Find $A(2v_1 - v_2 + 4v_3)$ and its length.