Parametric model

$f(x) = \lambda e^{-\lambda x}$ λ unknown

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$

μ, σ^2 unknown

$f(x) = f_\theta(x) = f(x;\theta) = f(x,\theta)$

make observations $\rightsquigarrow$ estimate θ / θ's $\rightsquigarrow$ more specific model, some confidence intervals, describe value of θ's.

Specific examples:

$\mu \rightsquigarrow \bar{X}$

$\sigma^2 \rightsquigarrow S^2$

Want: general procedures which take some general form of our distribution and unknown parameter $\rightsquigarrow$ estimators

This week:

- Method of moments
- Method of Maximum Likelihood.

Method of moments

Def k^{th} sample moment (about 0)

$$M_k' = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$ random var

X_i iid rand. vars $f_\theta(x)$

$$m_k' = \frac{1}{n}\sum_{i=1}^{n} x_i^k$$ ← observations.

Def $\mu_k' = E[X^k]$ X = rand var w/ dist. of pop

M_k' are unbiased estimators of μ_k'

Given exp. population: $f_\lambda(x) = \lambda e^{-\lambda x}$

Method of moments: solve for unknown parameters in terms of moments, soln gives estimators for params

$\mu = \frac{1}{\lambda}$ know $\mu_1' = \mu$

have $M_1' = \overline{X}$ estimator for μ

$\Rightarrow \frac{1}{M_1'} = \frac{1}{\overline{X}}$ estimator for λ

biased.

$f_\lambda(x) = \lambda e^{-\lambda x}$ $\quad f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}$

$\theta = \frac{1}{\lambda}$ $\quad \mu = \theta$ $\quad \overline{X}$ estimator for θ ($= \mu$)

unbiased

Example:

Gamma variable:

$$f_{\alpha,\beta}(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \qquad x > 0$$

$\mu_1' = \alpha\beta \qquad \mu_2' = \alpha(\alpha+1)\beta^2$

$\mu_2' = E[X^2]$

estimators for α, β

$$m_1' = \frac{1}{n}\sum X_i \qquad m_2' = \frac{1}{n}\sum X_i^2$$

$$\mu_1' \qquad \mu_2'$$

$$\mu_1' = \alpha\beta \quad \alpha = \frac{\mu_1'}{\beta} \rightsquigarrow \mu_2' = \alpha(\alpha+1)\beta^2$$

$$\mu_2' = \frac{\mu_1'}{\beta}\left(\frac{\mu_1'}{\beta}+1\right)\beta^2$$

$$\mu_2' = \left(\frac{(\mu_1')^2}{\beta^2} + \frac{\mu_1'}{\beta}\right)\beta^2$$

$$\mu_2' = (\mu_1')^2 + \mu_1'\beta$$

$$\beta = \frac{\mu_2' - (\mu_1')^2}{\mu_1'}$$

similarly $\alpha = \dfrac{(\mu_1')^2}{(\mu_2') - (\mu_1')^2}$

$$\hat{\beta} = \frac{M_2' - (M_1')^2}{M_1'} \qquad \hat{\alpha} = \frac{(M_1')^2}{M_2' - (M_1')^2}$$

Fun side topic:

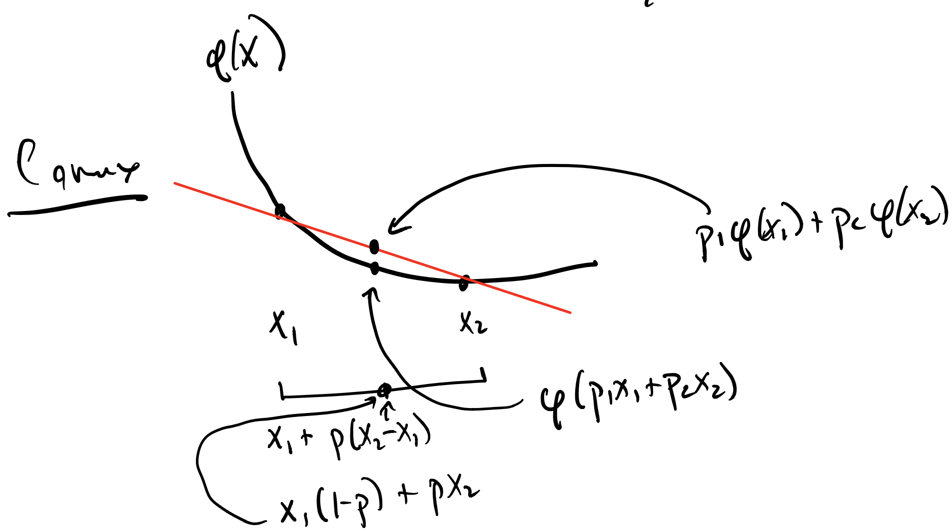
Why was $\frac{1}{\bar{X}}$ biased estimator for $\lambda = \frac{1}{\mu}$

Jensen's inequality

X is a random variable, φ convex function then

$$\varphi(E(X)) \le E[\varphi(X)]$$

with equality if φ is linear or in case φ is strictly convex, only if $\mathrm{Var}(X) = 0$.



$x_1 p_1 + x_2 p_2 \quad p_1 + p_2 = 1$

Def φ (strictly) convex if for $x_1 < x_2$

$$\varphi(p_1 x_1 + p_2 x_2) \le p_1 \varphi(x_1) + p_2 \varphi(x_2)$$

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