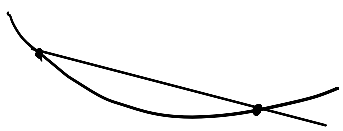
if φ convex X random var $\Longrightarrow$

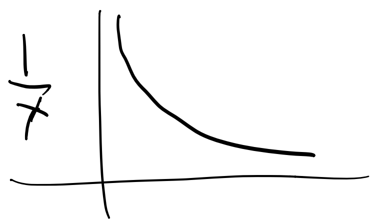
$$\varphi(E[X]) \leq E[\varphi(X)]$$

if φ strictly convex & $\operatorname{Var} X \neq 0$

$\Downarrow$

$$\varphi(E[X]) < E[\varphi(X)]$$





if $\hat{\theta}$ unbiased estimator for θ

$\Rightarrow$ often $\varphi(\hat{\theta})$ not unbiased for $\varphi(\theta)$

(particularly if φ is convex)

know $\bar{X}$ is an unbiased estimator μ

$$\Rightarrow \quad E\left[\frac{1}{\bar{X}}\right] > \frac{1}{E[\bar{X}]} = \frac{1}{\mu}$$

$\neq$

Consider case X finite $x_1, \dots, x_n$ finite set of values w/ prob. $p_i = p(x_i)$

will show: φ convex

$$\varphi(E[X]) \leq E[\varphi(X)]$$

if φ is strictly convex then $\varphi(E[X]) < E[\varphi(X)]$

Def φ convex $\Leftrightarrow$ for x_1, x_2 $\quad x = x_1 + p(x_2 - x_1)$

$0 \le p \le 1$

then $\varphi(x) \le \varphi(x_1) + p(\varphi(x_2) - \varphi(x_1))$

$\varphi(x_1 + p(x_2 - x_1))$

$p_2 = p \quad p_1 = (1-p)$

$x_1 + p(x_2 - x_1) = p_1 x_1 + p_2 x_2$

$p_1 \varphi(x_1) + p_2 \varphi(x_2)$

Def of concavity: X is a random var w/ 2 values x_1, x_2 prob p_1, p_2

then φ convex if for all

we have $\varphi(p_1 x_1 + p_2 x_2) \le p_1 \varphi(x_1) + p_2 \varphi(x_2)$

$E[X]$ $\quad E[\varphi(X)]$

φ convex if for X random var w/ values x_1, x_2

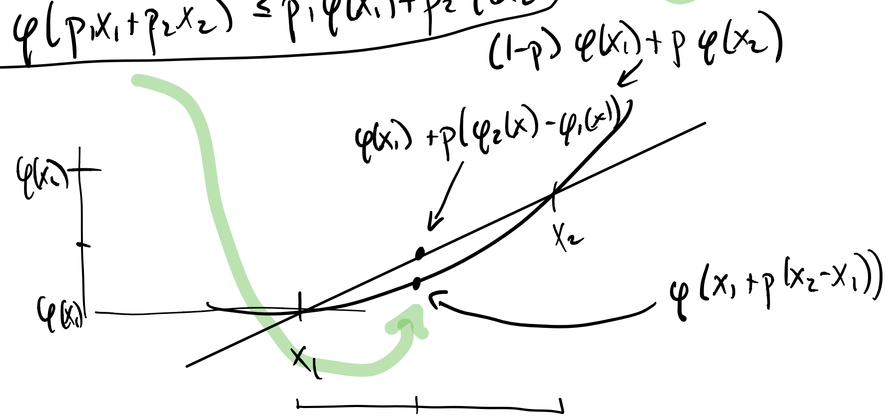
$\varphi(E[X]) \le E[\varphi(X)]$

Jensen's inequality

If X is a random variable, φ convex fcn

then $\varphi(E[X]) \le E[\varphi(X)]$

Def A function $\varphi(x)$ is convex if for x_1, x_2,
we have for $0 \le p_1, p_2 \le 1$, $p_1 + p_2 = 1$
then $\varphi(p_1 x_1 + p_2 x_2) \le p_1 \varphi(x_1) + p_2 \varphi(x_2)$



$p_1 = (1-p)$

$p_2 = p$

$x_1 + p(x_2 - x_1)$

$x_1 + p x_2 - p x_1$

$(1-p) x_1 + p x_2$

$0 \le p \le 1$

In fact

Prop $\varphi(x)$ is convex $\Longleftrightarrow$ given $x_1, \dots, x_n$;
$0 \le p_1, \dots, p_n \le 1$ $\quad \sum p_i = 1$ then

$$\varphi\left(\sum p_i x_i\right) \le \sum p_i \varphi(x_i)$$

Pf: $\Longleftarrow$ $n=2$

$\Longrightarrow$ prove by induction on n. $n=2$ def

induction step:

$$\varphi\left(\sum_{i=1}^n p_i x_i\right) = \varphi\left(p_1 x_1 + \sum_{i=2}^n p_i x_i\right)$$

$$= \varphi\left(p_1 x_1 + (1-p_1) \sum_{i=2}^n \left(\frac{p_i}{1-p_1}\right) x_i\right)$$

$$\sum_{i=2}^n p_i = 1 - p_1 \qquad \sum_{i=2}^n \frac{p_i}{1-p_1} = 1$$

$$\le p_1 \varphi(x_1) + (1-p_1) \varphi\left(\sum_{i=2}^n \left(\frac{p_i}{1-p_1}\right) x_i\right)$$

as convex

$$\le p_1 \varphi(x_1) + (1-p_1)\left(\sum_{i=2}^n \frac{p_i}{1-p_1} \varphi(x_i)\right)$$

induction

$$= p_1 \varphi(x_1) + \sum_{i=2}^n p_i \varphi(x_i) = \sum p_i \varphi(x_i) \; \square$$

Note: says: if $X \longleftrightarrow$ prob. fcn $p(x)$ (discrete)

$$\varphi(E[X]) = \varphi\left(\sum x_i p_i\right) \leq \sum p_i \varphi(x_i) = E[\varphi(X)]$$

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