Previous strategy

Bernoulli process, unknown parameter θ = prob of success

Do 7 trials, 6 success, 1 fail

$\exists$ random vars whose dist. is related to θ, so can use them probabilistically

what can we do?

estimate $\hat{\Theta} = \overline{X} \Rightarrow \theta \approx \frac{6}{7}$

$P(\theta) > \frac{1}{2}$?

can't ask this. θ is not a rand. var — it's a number.

this is a good estimator — min'l variance unbiased.

confidence interval?

e.g. find estimators $\hat{\Theta}_{sm}$, $\hat{\Theta}_{bg}$ s.t.

$$P(\hat{\Theta}_{sm} < \theta < \hat{\Theta}_{bg}) \geq .90$$

prob. statement.

after measurement, either true or false, not a prob. statement.

rough approx: $\sum X_i = n\overline{X}$ is binomial w/ mean $n\theta$ and var $n\theta(1-\theta)$

So

$$\frac{n\overline{X} - n\theta}{\sqrt{n\theta(1-\theta)}} \approx Z$$

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$$\sqrt{n}\,\frac{\bar{X}-\theta}{\sqrt{\theta(1-\theta)}}$$

P = .5 that

so, say $$\sqrt{7}\,\frac{\bar{X}-\theta}{\sqrt{\theta(1-\theta)}} \geq 2$$

$$\sqrt{7}(\bar{X}-\theta) \geq \sqrt{\theta(1-\theta)}$$

$$\sqrt{7}\,\bar{X} \geq \sqrt{\theta(1-\theta)} + \sqrt{7}\,\theta$$

graph?

$$\bar{X} \geq \sqrt{\frac{\theta(1-\theta)}{7}} + \theta$$

Bayesian Approach

Notice in previous case, we knew nothing about process beforehand. no info prior.

<u>Imagine</u>: have a bag of unfair coins,

equally likely to get one of any prob $\theta \in [0,1]$

we get one out.

this reps a population (of flips)

with an unknown parameter θ.

but θ is not as unknown as before.

previously, couldn't say $P(\theta > 1/2) = ?$

but now we can.

know $f(X \mid \theta) = \theta$ and $g(\theta) = \begin{cases} 1 & \theta \in [0,1] \\ 0 & \text{else} \end{cases}$

"$f(X \mid \Theta = \theta)$"

we can consider a "joint distribution"

$$f(x, \theta) = f(x \mid \Theta = \theta)$$

what if we get a success w/ one trial?

get a new (posterior) dist for θ

$$g(\theta \mid X=1)$$

via conditional parameter dist

$$g(\theta|x) = g(\theta \mid X=x) = \frac{f(x,\theta)}{f(x)} = f(x|\theta)\,g(\theta)$$

$\leftarrow$ marginal dist

$$f(x) = \int f(x,\theta)\,d\theta$$

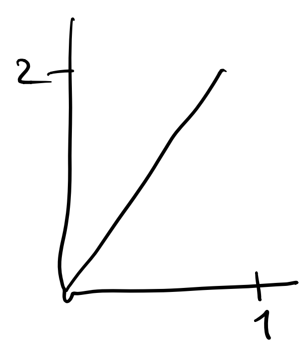
$$= \int f(x|\theta)\,g(\theta)\,d\theta$$

For example, in our case:

$$g(\theta \mid X=1) = \frac{f(x\mid\theta)\, g(\theta)}{\int f(x\mid\theta)\, g(\theta)\, d\theta}$$

$$= \begin{cases} \dfrac{\theta \cdot 1}{\int_0^1 \theta \, d\theta} & \theta \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 2\theta & \theta \in [0,1] \\ 0 & \text{else} \end{cases}$$



Straightforward in principle,

in practice, either need to do numerically

or leverage fcns w/ nice answers.

note: in general, $X \rightsquigarrow X_1, \ldots, X_n$!

Nice answer?

Start w/ some dist. $f(x|\theta)$

and an a priori dist $g(\theta)$ for θ.

now, assume the a priori dist. has some particular form

$g(\theta) = g(\theta, \varphi)$ w/ param. φ

φ <u>specified</u>

note, this is a "known" dist.

nice means given our sample,

$X_1, \ldots, X_n = x_1, \ldots, x_n$, the a posteriori dist

$g(\theta | x_1, \ldots, x_n)$ ~~is~~ has the form

$g(\theta, \varphi')$
some φ'.

Ex:
if $f(x,\mu)$ normal w/ known σ^2, unknown μ.

can consider an a priori dist for μ, $N(\mu_0, \sigma_0^2)$

$g(\mu)$ for μ via a random var M

Can then consider $g(\mu \mid \bar{X} = \bar{x})$

get another normal variable

of type $N\left(\frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2}, \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2} \right)$

Coin flip example?

if you keep flipping $n>1$, the a posteriori dist for θ is a Beta dist!

recall this is

$$g(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$x \in [0,1]$

"

$g(\theta \mid \alpha, \beta)$

" $\hat{\varphi}$ "

for $\alpha = 1, \beta = 1$ this is constant

$g(\theta) = 1.$

for $\alpha = 2, \beta = 1$ get $g(\theta) = 2x$

a posteriori for $\overline{X} = 1$, $n = 1$

and in general, a posteriori for $\overline{X} = \overline{x}$

so $n\overline{x}$ successes

" k

get
$$\boxed{\begin{aligned} \alpha &= k+1 \\ \beta &= n-k+1 \end{aligned}}$$
!

$\hat{\varphi}$

$$\sqrt{z}\,(\bar{X}-\theta) \geq \sqrt{\theta(1-\theta)}$$

$$x-y = \sqrt{y(1-y)}$$

$$(x-y)^2 = y(1-y)$$

$$x^2-2xy+y^2 = y(1-y)$$

$$= y-y^2$$

$$x^2-2xy+2y^2-y=0$$

$$x^2-2xy+2y^2-yz=0$$

$y=0 \quad z=1 \quad x=1$

project.