How to find the most powerful test?

Given two simple hypotheses H_0, H_1

Design test — divide up possible outcomes

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n = R_0 \cup R_1$$

$x \in R_0 \Rightarrow$ accept H_0 reject H_1

$x \in R_1 \Rightarrow$ accept H_1 reject H_0

Convention: we call R_1 = "critical region" $= \mathcal{C}$

$P(x \in \mathcal{C} \mid H_0) = \alpha$ smaller than some fixed value. "type 1 errors"

$1-\beta = P(x \in \mathcal{C} \mid H_1)$ ← want as large as possible.

$\beta = P(x \notin \mathcal{C} \mid H_1)$ "type 2 error"

It is possible (reasonable) that for values $x \notin \mathcal{C}$ (discrete)

$$P(X = x \mid H_0) < P(X = x \mid H_1)$$

ex: H_0 = coin fair $H_1 = P(\text{heads}) = 80\%$

experiment: 5 flips. $\mathcal{C} = \{ 5 \text{ heads only} \}$

$P(\text{type 1 error}) = \frac{1}{32}$

$P(X=4 \mid H_0) = \binom{5}{1}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 = \frac{5}{32} \approx 16\%$

$P(X=4 \mid H_1) = \binom{5}{1}\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^1 = \frac{5}{5^5} \cdot 4^4 = \frac{2^8}{625}$

$= \frac{256}{625} \approx 40\%$

else: if allowed $C = \{4 \text{ or } 5 \text{ heads}\}$

$P(\text{type 1}) = P(X = 5 \text{ or } 4 \mid H_0)$

$= \frac{1}{32} + \frac{5}{32} = \frac{6}{32} = \frac{3}{16} \approx 19\%$

Its also totally feasible for $x \in C$

$P(X=x \mid H_0) > P(X=x \mid H_1)$

H_0 = fair coin $\quad H_1 = 99.5\%$ heads.

experiment: 10 flips, accept H_1 if 9 or 10 heads.

$P(X = 9 \text{ or } 10 \mid H_0) \approx 1\%$

check $\quad P(X=9 \mid H_0) \approx 1\%$

$P(X=9 \mid H_1) < \frac{1}{2}\%$

Goal: How to find the most powerful test w/ given α?

$\alpha = P(x \in C \mid H_0)$ "size of C"
"significance level of the test"

Def Fix a size α for our crit regions, only consider regions of size $\leq \alpha$.

We say that C is a most powerful region for hypotheses H_0, H_1 if for any other region D of size (of size $\leq \alpha$) $\leq \alpha$, we have $P(X \in C \mid H_1) \geq P(X \in D \mid H_1)$

$(1-\beta)$
"power"

Neyman–Pearson Lemma

Discrete case C crit region of size α

If $\dfrac{P(X = x \mid H_0)}{P(X = x \mid H_1)}$ is at least as big for $x \notin C$ as it is for all $x \in C$ then C is a most powerful region for α

assume $P(X=x \mid H_1) \neq 0$ all x...

(in region where we accept H_0 want this big
— — — — — H_1 — — small.)

& $P(X \in C \mid H_0) \neq 0$

more formally $\dfrac{P(X=x|H_0)}{P(X=x|H_1)} \geq \dfrac{P(X=y|H_0)}{P(X=y|H_1)}$

all $x \notin C$, $y \in C$.

<u>Proof</u>

set $K = \inf \left\{ \dfrac{P(X=x|H_0)}{P(X=x|H_1)} \,\middle|\, x \notin C \right\}$

above inequality $\Rightarrow \dfrac{P(X=y|H_0)}{P(X=y|H_1)} \leq K$

$K \neq 0$ $\dfrac{P(X=x|H_0)}{P(X=x|H_1)} \geq K$

$\Rightarrow$

$$\Rightarrow \frac{P(X=x|H_0)}{K} \geq P(X=x|H_1) \quad x \notin C$$

$$\frac{P(X=y|H_0)}{K} \leq P(X=y|H_1) \quad y \in C$$

Suppose D is a region of size $\leq \alpha$

want $P(X \in C|H_1) \geq P(X \in D|H_1)$

Know: $P(X\in C|H_0)=\alpha \geq P(X\in D|H_0)$

$$P(X\in C|H_0) = P(X\in \cancel{C\cap D}|H_0)+P(X\in C\cap D^c|H_0)$$

$$P(X\in D|H_0) = P(X\in \cancel{D\cap C}|H_0)+P(X\in D\cap C^c|H_0)$$

$$\frac{P(X\in C\cap D^c|H_0)}{k} \geq \frac{P(X\in D\cap C^c|H_0)}{k}$$

$$\geq \qquad\qquad\qquad\qquad \geq$$

$$\underbrace{P(X\in C\cap D^c|H_1)} \geq \underbrace{P(X\in D\cap C^c|H_1)}$$

$$P(X\in C|H_1)$$

$$= P(X\in C\cap D|H_1)+P(X\in C\cap D^c|H_1)$$

$$\geq P(X\in C\cap D|H_1)+P(X\in D\cap C^c|H_1)$$

$$= P(X\in D|H_1) \quad \square$$

Neyman Pearson (cont case)

heuristically
replace $P(X=x|H_0)$ by $f_0(x)$ pdf H_0
& $P(X=x|H_1)$ by $f_1(x)$ pdf H_1

want to say: if $\dfrac{f_0(x)}{f_1(x)} \geq \dfrac{f_0(y)}{f_1(y)}$ $\Rightarrow C$ most powerful

$f_1(x) \neq 0 \ldots$ $\quad x \notin C, \; y \in C$

Actual statement: If C has pow α, and

If $\exists L$ s.t. $L f_0(x) \geq f_1(x)$ $\;x \notin C$

$L f_0(y) \leq f_1(y)$ $\;y \in C$

then C is most powerful for α.

$$K = \inf\left\{ \frac{f_0(x)}{f_1(x)} \,\middle|\, x \notin C \right\} \rightsquigarrow L = \frac{1}{K}$$

Pf in cont case

$$P(x \in C|H_0) = \alpha \geq P(x \in D|H_0)$$

$\quad\;\; \parallel \qquad\qquad\qquad\qquad \parallel$

$$\int_{x\in C} f_0(x)\,dx = \int_C f_0$$

$$\int_{x\in D} f_0(x)\,dx = \int_D f_0$$

$$\int \cdot \int\int \cdots \int f_0(x_1,\dots,x_n)\,dx_1\cdots dx_n$$

$$= \prod \tilde{f}_0(x_i)$$

$$\int_C f_0 = \alpha \geq \int_D f_0$$

$$\int_C f_0 = \cancel{\int_{C\cap D} f_0} + \int_{C\cap D^c} f_0$$

$$\int_D f_0 = \cancel{\int_{D\cap C} f_0} + \int_{D\cap C^c} f_0$$

$$\int_{C\cap D^c} f_0 \geq \int_{D\cap C^c} f_0$$

$$1-\beta = \int_C f_1 = \int_{C\cap D} f_1 + \int_{C\cap D^c} f_1 \geq \int_{C\cap D} f_1 + L\int_{C\cap D^c} f_0 \geq L\int_{D\cap C^c} f_0 + \int_{C\cap D} f_1$$

$$\geq \int_{D\cap C^c} f_1 + \int_{C\cap D} f_1 = \int_D f_1$$

$$\geq$$