normally distributed

$X_1, \ldots, X_n \qquad \mu, \sigma^2$

knew $\bar{X}$ normally dist.

$\rightsquigarrow \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ ← standard normal variable.

S^2 = sample variance

turns out $\dfrac{(n-1)S^2}{\sigma^2}$ has a χ^2_{n-1}

"chi-square distribution w/ n-1 degrees of freedom"

χ^2_{n-1} is a particular Γ variable

$\alpha = \frac{n}{2} \quad \beta = 2$

MGF $(1-2t)^{-\frac{n}{2}}$

<u>Example</u> Checking length of widgets (Assume lengths of widgets is normally distributed)

$n=10$ find $S^2 = 5$

<u>Q</u>: how likely is it that the actual variance $\sigma^2 \geq 2$?

Know: $\chi^2_9 = 9S^2/\sigma^2$

$$P\left(\sigma^2 \geq \frac{2}{5} S^2\right)$$

$$= P\left(\frac{5}{2} \geq \frac{S^2}{\sigma^2}\right)$$

$$= P\left(\frac{9 \cdot 5}{2} \geq \frac{9S^2}{\sigma^2}\right)$$

$$= P\left(22.5 \geq \chi^2_9\right)$$

$$\approx .995$$

99.5 % chance that pop variance is $\geq \frac{2}{5}$ (sample variance)

What is χ^2_m?

Γ random variable w/ $\alpha = \frac{n}{2}$ $\beta = 2$

Gamma variable

pdf: $$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Why is χ^2_m relevant?

if Z is a stand. norm rand variable

$\Rightarrow Z^2$ is a Gamma variable!

$$Z^2 = \chi^2_1$$

$\Rightarrow$ if $Z_1, \dots, Z_m$ are iid std normal then

$$\sum Z_i^2 = \chi^2_m$$

$\boxed{\text{Sample variance \& Chi-square}}$

$$\frac{(n-1)S^2}{\sigma^2} = \sum \left(\frac{X_i - \bar{X}}{\sigma} \right)^2$$

$$\boxed{S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1}}$$

$$= \sum \left(\frac{(X_i - \mu) + (\mu - \bar{X})}{\sigma} \right)^2$$

$$= \sum_{i=1}^{n} \left[\frac{(X_i - \mu)^2 + 2(X_i - \mu)(\mu - \bar{X}) + (\mu - \bar{X})^2}{\sigma^2} \right]$$

$$= \left[\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^2}\right] + \left[\frac{2(\mu-\bar{X})}{\sigma^2} \underbrace{\sum_{i=1}^{n}(X_i-\mu)}\right] + \frac{(\bar{X}-\mu)^2}{\sigma^2}n$$

$$\boxed{\begin{gathered}\frac{\sum X_i}{n} = \bar{X} \\ \sum X_i = n\bar{X}\end{gathered}}$$

$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - \frac{2(\bar{X}-\mu)}{\sigma^2}\left(n\bar{X}-n\mu\right) + \left(\frac{\bar{X}-\mu}{\sigma}\right)^2 n$$

$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - 2n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2 + n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 = \frac{(n-1)S^2}{\sigma^2}$$

std. normals

std normal

$$\rightsquigarrow \quad \underbrace{\frac{(n-1)S^2}{\sigma^2}}_{Y} + \underbrace{\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2_1} = \underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2}_{\chi^2_n}$$

If $\quad$ // are independent, MGF of sum

$= \prod$ of MGF's.

They are independent.

$$M_y(t)\,M_{\chi_1^2}(t) = M_{\chi_n^2}(t)$$

$$M_y(t)\,(1-2t)^{-\frac{1}{2}} = (1-2t)^{-n/2}$$

$$M_y(t) = (1-2t)^{-\left(\frac{n-1}{2}\right)}$$

$= \text{MGF for } \chi^2_{n-1}$

$\Rightarrow y$ is a χ^2_{n-1} random var

$$\overset{\shortparallel}{\frac{(n-1)S^2}{\sigma^2}}.$$