Math 477, Practice sheet for Exam 1

This exam will cover sections 1.1–1.6, 2.1–2.5, 3.1–3.5, 4.1–4.7 and 4.8.1, 4.8.2.

The problems below do not necessarily cover all the topics completely, but will hopefully still be helpful as a reminder of some of the material covered. Please also look through prior worksheets and homework assignments.

1. Consider sequences of n numbers, each in the set $\{1, 2, \ldots, 6\}$.

   (a) How many sequences are there if each number in the sequence is distinct?

   Solution. There are various ways to write this, but one way is: $\binom{6}{n}n!$ (number of ways to choose n distinct numbers, and the number of ways to order them). □

   (b) How many sequences are there if no two consecutive numbers are equal?

   Solution. $(6)(5)^{n-1}$. □

   (c) How many sequences are there if 1 appears exactly i times in the sequence?

   Solution. $\binom{n}{i}5^{n-i}$. □

2. Suppose that a basket contains n red balls and m blue balls. The balls are removed from the basket, each one equally likely until r red balls have been removed. What is the probability that a total of k balls have been removed at this point?

   Solution. Let S be the set of all possible sequences consisting of n red and m blue balls. Let E be the set of those sequences such that the k'th ball in the sequence is the r'th ball chosen so far. We then need to find

   $$P(E) = \#E/\#S$$

   The elements of S are determined by specifying which of the n balls in the sequence of $n+m$ balls are red. Therefore $\#S = \binom{n+m}{n}$. The elements of E correspond to sequences where the k'th element in the sequence is red, where $r-1$ of the first $k-1$ elements are red, and where the remaining $n-r$ red balls are distributed among the last $n+m-k$ elements. The number of such sequences is therefore

   $$\#E = \binom{k-1}{r-1}\binom{n+m-k}{n-r}$$

   and so the probability is

   $$\frac{\binom{k-1}{r-1}\binom{n+m-k}{n-r}}{\binom{n+m}{n}}.$$

   □

3. Suppose that we have 2 coins, the first, when flipped, has a 90% heads and the second has a 50% chance of resulting in heads or tails. Suppose a coin is picked (each coin being equally likely). When flipped, the result is heads. What is the probability that, if flipped again, the result will be heads?

Solution. Let F be the event that the coin has a 1/2 chance of heads, and F^c the event that the coin has a 9/10 chance of heads. Let A be the event that the first flip is heads, and B the event that the second flip is heads. We want to calculate $P(B|A)$. By definition, we have $P(B|A) = P(AB)/P(A)$. We can then compute:

$$P(AB) = P(ABF) + P(ABF^c) = P(AB|F)P(F) + P(AB|F^c)P(F^c) = (9/10)^2(1/2) + (1/2)^2(1/2)$$

$$P(A) = P(AF) + P(AF^c) = P(A|F)P(F) + P(A|F^c)P(F^c) = (9/10)(1/2) + (1/2)(1/2)$$

and so

$$P(AB)/P(A) = \frac{(9/10)^2(1/2) + (1/2)^3}{(9/10)(1/2) + (1/2)^2}.$$

□

4. Suppose that we have 2 coins, the first, when flipped, has a 90% heads and the second has a 50% chance of resulting in heads or tails. Suppose a coin is picked (each coin being equally likely), and is flipped over and over until the result is tails. What is the expected number of flips this will take?

   Solution. I'm going to give some extra details in the solution here. There is more than one way to do this, but I'm going to do it by carefully going over various facts and definitions...

   Let Y be the random variable representing the number of flips needed assuming we have the first coin, and Z be the random variable representing the number of flips needed assuming we have the second coin. Notice that Y and Z are geometric random variables with $p = 9/10$ or $p = 1/2$. In particular for the first coin, we have $P(Y = i) = (9/10)^{i-1}(1/10)$ for the probability that the coin results in a tails on the ith flip, and for the second, a probability of $P(Z = i) = (1/2)^{i-1}(1/2) = (1/2)^i$ that we get a tails on the ith flip.

   Now for the actual problem. Let X be the random variable denoting the number of flips needed for the unknown coin. Let F denote the event that the coin has a 50% chance of heads or tails, and so F^c is the event that the coin has a 90% chance of heads. We know that $P(F) = 1/2 = P(F^c)$. Let H_i, $i = \{1, 2, \ldots\}$, be the event that the coin results in heads on the i'th flip, so that $P(X = i) = H_i$. By the previous discussion, we then have $P(H_i|F^c) = P(Y = i) = (9/10)^{i-1}(1/10)$ and $P(H|F) = P(Z = i) = (1/2)^i$. We therefore have

   $$P(X = i) = P(H_i) = P(H_i|F)P(F) + P(H_i|F^c)P(F^c) = P(Y = i)(1/2) + P(Z = i)(1/2)$$

   We therefore have

   $$E[X] = \sum_i iP(X = i) = \sum_i i\left(P(Y = i)(1/2) + P(Z = i)(1/2)\right) =$$
   $$1/2\sum_i iP(Y = i) + 1/2\sum_i P(Z = i) = 1/2E[Y] + 1/2E[Z]$$

   Since Y and Z are geometric variables with $p = 1/10, 1/2$ respectively, we have $E[Y] = 10/9$ and $E[X] = 2$. Therefore $E[X] = 1/2(10/9 + 1/2) = 28/18 = 14/9$. □

5. Two dice are rolled, and the results are added. Assuming that this number is greater than or equal to 8, what is the probability that one of the dice rolled a 6?

6. In the game "raven's beak," a player rolls 6 dice, and wins if at least three of the dice roll the same number. What is the probability of winning?

7. In the game "dove's gambit," a player rolls 6 dice, and wins if at least three of the dice roll the number 1. What is the expected number of games played before the player wins?

8. A swarm of flying insects are flying around a lamp. If there are 1000 insects, and each insect has a probability of $1/2000$ of bumping into the light every second, estimate the probability that no more than 3 insects hit the lamp after 10 seconds?

Solution. This is really asking to estimate the probability $P(X \leq 3)$ where X is the number of insects which hit the lamp after 10 seconds. One first approximation for this would be to say that each of the 1000 insects takes 10 tries to hit the lamp, which would give a total of 10,000 tries, each with a probability of $p = 1/2000$ of success. If we ignore the fact that it is possible that the same insect can hit the lamp twice (which is not very likely), this would be given by a binomial variable with $n = 10000$ and $p = 1/2000$. On the other hand, since $\lambda = pn = 10$ is small compared to $n = 10000$, we see that we can approximate X with a Poisson variable with $\lambda = 10$. $\square$

9. A swarm of flying insects are flying around a lamp. If there are 1000 insects, and each pair of insects has a probability of 1/1,000,000 of bumping into each other every second, estimate the expected number of collisions after 1 minute?

Outline of solution. Like we did in the birthday problem, we will estimate this by saying that there are $\binom{1000}{2}$ pairs of insects, and each pair bumps at an average rate of once per 1,000,000 seconds. We can then estimate the number of bumps for all pairs per second, and then muliply by 60 to get the expected number of bumps per minute. We are then done.

While it is then true that we will expect the number X of bumps in 1 minute to satisfy a Poisson distribution, we don't need to do anything with this information to answer the question! This would have been relevant, however, if we were asked the probability that there would be some particular number of collisions, however... $\square$