textbook "A first course in probability"
by Sheldon Ross (any edition)

Part 1: Counting (strategies)
"Combinatorics" "Combinatorial analysis"

"Addition principle" if A & B are disjoint sets
then $\#A + \#B = \#(A \cup B)$

$\#A = |A| =$ order of A

"Multiplication principle" i.e. the basic principle of counting

$$\#(A \times B) = (\#A)(\#B)$$

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

if there are n ways to choose a

& if there are m ways to choose b (for any way to choose a)

then there are $n+m$ ways to choose either a or b

and there are nm ways to choose both a & b

---

example: given deck of 52 cards, want to choose 3 distinct cards in a sequence

i.e. card 1 = 52
card 2 = 51
card 3 = 50

$\#$ways $= 52 \cdot 51 \cdot 50$

how many ways to arrange all cards in a sequence?

$52 \cdot 51 \cdot 50 \cdot 49 \cdot \ldots\ldots 1 = 52!$

Division principle: Given two sets A, B,

if for each thing in set B, there are n things in set A then $\#A = n\#B$.

More formally, if we have a function $A \xrightarrow{f} B$

s.t. for each $b \in B$, $\#f^{-1}(b) = n$

then $\dfrac{\#A}{n} = \#B$

$$\begin{pmatrix}\text{Choice of sequence} \\ \text{of 3 cards}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{unordered} \\ \text{hand of 3 cards}\end{pmatrix}$$

1: A♡
2: 10◇
3: 3◇

$\longrightarrow$ {A♡, 3◇, 10◇}

6 things $\longrightarrow$ 1 thing

$\#$ ordered hands $= 6\ \#$ unordered hands

$$\#\text{ordered hands} = 52 \cdot 51 \cdot 50$$

$$\#\text{unordered hands} = \frac{52 \cdot 51 \cdot 50}{6}$$

General pattern

Given n distinct things, how many ways can we choose exactly k of them?

if the k things are ordered in a sequence

$$n-(k-1) = n-k+1$$

sequence $\underbrace{\ }_{1}^{n}\ \underbrace{\ }_{2}^{n-1}\ \underbrace{\ }_{3}^{n-2}\ \underbrace{\ }^{n-3}\ \cdots\ \underbrace{\ }_{k}^{n-k+1}$

$$\boxed{n(n-1)(n-2)\cdots(n-k+1)} = \frac{n!}{(n-k)!}$$

if unordered

(ordered sequence of k) $\longrightarrow$ (unordered seq of k)

all the ways of ordering it $= k!$ things $\rightsquigarrow$ same unordered seq

$$\Rightarrow \text{\# of unordered seq. of } k \text{ from } n = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k,\,n-k}$$

$$\binom{5}{2} = \binom{5}{3}$$

$$\binom{5}{2,\,3}$$

---

flip a coin 10 times, how many ways can we do this and get exactly 3 heads?

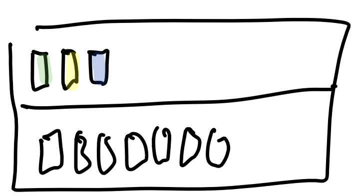
(out of 2^{10} total flip sequences)

answer: $\binom{10}{3}$

---

10 books, distribute them into 2 shelves

shelf 1 has 3 books, shelf 2 has 7?



$\binom{10}{3,7}$ = # ways to choose top vs bottom shelf for each book.

order matters $\Rightarrow$ mult. by $3!$ (top)

$7!$ (bottom)

answer $\binom{10}{3,7} \cdot 3!\,7! = \frac{10!}{3!\,7!} \cdot 3!\,7! = 10!$

2 teams of 10 people each

how many ways can we pair off one person from each team.

A1 A2 - - A10

B1 B2 - - B10

$\boxed{10!}$