Warm up: Binomial

Binomial (p,n)

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} \quad i=0,1,\dots,n$$

will compute expectation & variance

Functions of a random variable

X random variable $(X: S \to \mathbb{R})$

given any other function $f: \mathbb{R} \to \mathbb{R}$

can define a new random variable $f(X)$

via $f(X): S \xrightarrow{X} \mathbb{R} \xrightarrow{f} \mathbb{R}$

$$f(X)(s) \equiv f(X(s))$$

ex: X = result of die throw $X = \{1,2,\dots 6\}$

$$X^2 = \{1,4,9,\dots,36\}$$

$$E[X] = \sum_{s \in S} p(s) X(s) = \sum_i P(X=a_i)\, a_i$$

if sample space S is countable $\quad X = a_1, a_2, \dots$

$$E[f(X)] = \sum_{s \in S} p(s) f(X(s)) \underset{\text{check}}{=} \sum_i P(X=a_i) f(a_i)$$

Applications:

$$E[\lambda X] = \sum_s P(s)\,\lambda X(s) = \lambda \sum_s P(s)\,X(s)$$
$$= \lambda E[X]$$

$$E[X+\mu] = E[X] + \mu$$

Recall: $\mathrm{Var}(X) = E[(X-\mu)^2]$

$$\mu = E[X]$$

$$\mathrm{Var}(X) = E[X^2 - 2X\mu + \mu^2]$$
$$= E[X^2] - E[2X\mu] + E[\mu^2]$$
$$= E(X^2) - 2\mu E[X] + \mu^2$$
$$= E[X^2] - 2\mu\cdot\mu + \mu^2 = E[X^2] - \mu^2$$
$$= E[X^2] - E[X]^2$$

X = die toss:

$$E[X] = \frac{1}{6}(1+2+\cdots+6) = \frac{21}{6} = \frac{7}{2}$$

$$E[X^2] = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{91}{6} - \frac{49}{4}$$
$$\frac{182}{12} - \frac{147}{12} = \frac{35}{12}$$

$X = \text{binomial}(p, n)$

$$E[X^k] = \sum_{i=0}^{n} i^k p(X=i) = \sum_{i=1}^{n} i^k P(X=i)$$

$$= \sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

$$i\binom{n}{i} = \frac{n!}{i!(n-i)!} i$$

$$= \frac{n!}{(i-1)!(n-i)!}$$

$$= n\frac{(n-1)!}{(i-1)!(n-i)!} = n\binom{n-1}{i-1}$$

$$= \sum_{i=1}^{n} i^{k-1} n \binom{n-1}{i-1} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

let $j = i-1$ $(i = j+1)$

$$np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-\cancel{(i-1)}\,j}$$

Y binom $(n-1, p)$

$$= np\, E[(Y+1)^{k-1}]$$

$k=1$ $\quad E[X] = np\, E[(Y+1)^0] = np$

$(n-1)p$

$k=2$ $\quad E[X^2] = np\, E[Y+1] = np\, E[Y] + np\, E[1]$

$$= np(n-1)p + np$$

$$= (n^2 p - np)p + np = n^2p^2 - np^2 + np$$