Chapter 5: Continuous random variables

Today: Introduce concepts, important distributions
(5.1 – 5.5)

Continuous random variables

We say a random variable X is continuous with probability density function $f(x)$ if

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

$$\left(\text{more generally } P(X \in B) = \int_B f(x)\,dx \right)$$

(by comparison, if X discrete (countable)

$$P(X \in B) = \sum_{x \in B} P(X = x)$$
)

Def If X is a cont. random variable w/ density fcn $f(x)$, We define the cumulate distribution function $F(x)$ of X to be $F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$

$\left(\text{compare to } \underset{\text{discrete}}{P(X \le a)} = \sum_{x \le a} P(X = a) \right)$

$$\text{FTC} \Rightarrow \frac{d}{da} \int_{-\infty}^{a} f(x)\,dx = \frac{d}{da} F(a) = f(a)$$

Notice: If X is a continuous random variable

$$P(X = a) = 0 = \int_{a}^{a} f(x)\,dx$$

ex: roll a ball across the floor X = distance when it stops

ex: swarm of insects around lamp X = time until first collision.

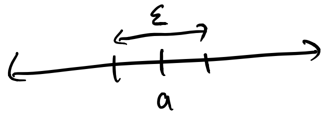
Observation: If X is a cont. rand. variable w/ density $f(x)$

then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$ and $\underset{\text{all } x.}{f(x) \ge 0}$

$$\text{"} P(X \in \mathbb{R})$$

Intuition: $f(a) \longleftrightarrow$ prob that X is close to a (proportional to)

$$P\left(|X - a| \le \frac{\varepsilon}{2} \right) = \int_{a - \frac{\varepsilon}{2}}^{a + \frac{\varepsilon}{2}} f(x)\,dx \approx f(a) \cdot \varepsilon$$

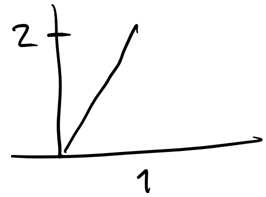


Expectation

Recall in discrete case $E[X] = \sum_x x P(X=x)$

X continuous $\longrightarrow$ $\boxed{E[X] \equiv \int_{-\infty}^{\infty} x f(x)\,dx}$

(if we think of $f(x)$ as "density" or mass distribution then $E[X]$ = center of mass)



$$f(x) = \begin{cases} 2x & x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big]_0^1 = \frac{2}{3}$$

Variance: As before: $\operatorname{Var}(X) = E\left[(X - E[X])^2\right]$

Set $\mu = E[X]$ $\qquad \operatorname{Var}(X) = E\left[(X-\mu)^2\right]$

As before, one can check: $\operatorname{Var}(X) = E[X^2] - E[X]^2$

One can consider functions of random variables

$g(x)$ some fcn X random var. $Y = g(X)$

$Y = 7X$ if $X \leftrightarrow$ density fcn f_X cum d.f. F_X

$Y \longleftrightarrow \cdots$ f_Y cdf F_Y

$$F_Y(a) = P(Y \le a) = P(7X \le a)$$
$$= P\left(X \le \tfrac{a}{7}\right)$$
$$= F_X(a/7)$$

$$f_Y(a) = \frac{d}{da} F_Y(a) = \frac{d}{da} F_X(a/7) = \frac{1}{7} f_X(a/7)$$

$$f_Y(x) = \frac{1}{7} f_X(x/7)$$

5.1-5.2

Some Important Continuous distributions

- Uniform 5.3
- Normal 5.3, 5.4
- Exponential 5.5

Def We say a continuous random variable is <u>uniformly</u> distributed on an interval (α,β) if its prob. density function is constant on (α,β) and 0 outside (α,β)

in particular: $$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in (\alpha,\beta) \\ 0 & \text{else.} \end{cases}$$

Def We say a continuous random variable is <u>normally</u> distributed w/ parameters μ, σ^2 if its density function is given as

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad x \in \mathbb{R}$$

this naturally occurs as a limit of binomial variables:

if Y_n = binomial w/ params p, n

$$\mu_n = E[Y_n] = pn$$

$$\sigma_n^2 = Var(Y_n) = pn(1-p)$$

$$P\left(a \le \frac{Y_n - \mu_n}{\sigma_n} \le b\right) \xrightarrow[n\to\infty]{} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

Def A continuous random variable X is exponentially distributed w/ parameter λ if its prob. density is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

if Y is Poisson w/ parameter λ = avg. # of occurrences per time interval

$P(Y=n) \longleftrightarrow$ "prob. that process happens n times in the interval"

interpretation

P(first occurrence happens within k units of time)

$= 1 - P(\text{no occurrences in } k \text{ units of time})$

$= 1 - P(\text{no occurrences in } 1 \text{ unit})^k$

$= 1 - e^{-k\lambda}$

$P(Y=0)$

$" \frac{1}{0!} e^{-\lambda}$

X = time to first occurrence

$F(k) = P(X \leq k) = 1 - e^{-k\lambda}$

$$F(x) = 1 - e^{-x\lambda}$$

$$\Rightarrow f(x) = \frac{d}{dx} F(x) = \lambda e^{-x\lambda}$$