Warmup:

X & Y random variables w/ joint density

$$f(x,y) = \begin{cases} x+y & x, y \in [0,1] \\ 0 & \text{else} \end{cases}$$

a) are X & Y independent? why or why not?

b) find $f_X(x)$ (pdf for X)

c) find $P(X+Y<1)$

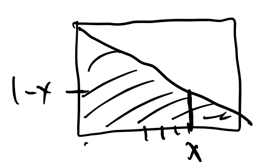
$$f_X(x) = \int_0^1 (x+y)\,dy = x\int_0^1 1\,dy + \int_0^1 y\,dy$$

$$= x\,[y]_0^1 + \left[\tfrac{1}{2}y^2\right]_0^1$$

$$= x \cdot 1 + \left(\tfrac{1}{2} - 0\right) = x + \tfrac{1}{2}$$

$f_Y(y) = y + \frac{1}{2}$ not independent.

c. $$P(X+Y<1) = \int_{x=0}^{1} \int_{y=0}^{1-x} (x+y)\,dy\,dx$$



$$= \int_{x=0}^{1}\left[\int_{y=0}^{1-x} x\,dy + \int_{y=0}^{1-x} y\,dy\right]dx$$

$$= \int_{x=0}^{1} \left(x \left[y \right]_{y=0}^{1-x} + \left(\frac{1}{2} y^2 \right)_0^{1-x} \right) dx$$

$$= \int_0^1 \left(x(1-x) + \frac{1}{2}(1-x)^2 \right) dx$$

$$= \int_0^1 x - x^2 + \frac{1}{2}(1 - 2x + x^2) \, dx$$

$$= \int_0^1 \cancel{x} - x^2 + \frac{1}{2} - \cancel{x} + \frac{1}{2} x^2 \quad dx$$

$$\int_0^1 \left(\frac{1}{2} - \frac{1}{2} x^2 \right) dx$$

$$\frac{1}{2} \int_0^1 (1 - x^2) \, dx$$

$$\frac{1}{2} \left[1 - \frac{1}{3} \right] = \frac{1}{2} \frac{2}{3} = \frac{1}{3}$$

Sums of random variables

Reminder: If X & Y are independent random variables

Consider $X+Y$

if both are discrete, $p_X(a) = P(X=a)$

$$p_Y(b) = P(Y=b)$$

$$p_{X+Y}(c) = \sum_{\substack{a,b \\ a+b=c}} p_X(a)\, p_Y(b)$$

$$P(\underbrace{X+Y=c}) = P(E_c S) = P(E_c(\bigcup_a F_a))$$

$$E_c \qquad F_a \qquad S = \underbrace{\bigcup_a F_a}_{\text{disjoint}}$$

$$X+Y=c \qquad X=a$$

$$= P(\bigcup_a E_c F_a) = \sum_a P(E_c F_a)$$

$$P(E_c F_a) = P(X=a;\ X+Y=c)$$

$$= P(X=a;\ Y=c-a)$$

$$p_{X+Y}(c) = \boxed{\sum_{a} p_X(a)\,p_Y(c-a)} = \sum_{\substack{a,b \\ a+b=c}} p_X(a)\,p_Y(b)$$

if X & Y are continuous w/ p.d.f's f_X, f_Y, X, Y indep.

then $$f_{X+Y}(c) = \int_{t=-\infty}^{\infty} f_X(t)\,f_Y(c-t)\,dt$$

"convolution of f_X & f_Y"

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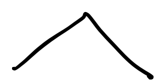
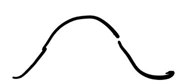
unif. random variables $X_1, X_2, \ldots$ identical, independent.

$X_1 + X_2$ 

$X_1 + X_2 + X_3$ 

$\vdots$

<u>exponential random vars</u>

<u>Gamma</u>.

<u>normal rand var.</u>

sums of normal

" normal

Recall: exponential ⟿ Poisson

prob. of occurrence after n tries (w/ geometric)

Gamma. ⟿ prob. of k occurring (neg binomial)

Recall: exponential: $f(x) = \lambda e^{-\lambda x}$ λ "avg rate."

$= Ce^{-\lambda x}$ $x \in [0, \infty]$

2 exponentials: $f(x) = Ce^{-\lambda x}$ $g(x) = Ce^{-\lambda x}$

X Y

Cookie for Timon

$$X + Y \rightsquigarrow h(x) = \int_0^\infty f(t) g(x-t)\, dt$$

$x - t > 0$

$t < x$

$$= C' \int_0^\infty e^{-\lambda t} \overbrace{e^{-\lambda(x-t)}} dt$$

$$= C' \int_0^x e^{-\lambda x} dt$$

$$C' e^{-\lambda x} \int_0^x dt = C' e^{-\lambda x} x$$

$X+Y+Z \rightsquigarrow C' e^{-\lambda x} x^2$

k identical indep as above $\rightsquigarrow C' e^{-\lambda x} x^k$

<u>Def</u> A cont. random variable has the <u>Gamma distribution</u>

if $f(x) = C e^{-\lambda x} x^{\alpha - 1}$ (w/ parameters (α, λ))

$x \geq 0$, 0 else $\quad \alpha > 0$

In this case we have:

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\Gamma(\alpha) =$ the thing that makes ($\int = 1$)

$$= \int_0^\infty e^{-y} y^{\alpha - 1} \, dy$$

$\Gamma(n) = (n-1)!$ $\quad$ Gamma var $\alpha = 1, \lambda$

$n \geq 1$ integer $\quad \Leftrightarrow$ exponential λ

<u>Important Property</u>: if X is Gamma w/ α, λ

Y - - - - w/ β, λ

then $X+Y$ is Gamma w/ $\alpha+\beta, \lambda$

Sums of Independent Normals

$X \qquad f(x) = Ce^{-(x-\mu)^2/2\sigma^2}$

if X normal μ_X, σ_X^2

Y - -- μ_Y, σ_Y^2

Then $X+Y$ normal w/ $\mu_X+\mu_Y$, $\sigma_X^2+\sigma_Y^2$