X random var w/ p.d.f. $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$

Y uniform on $[0,L]$ $\longleftrightarrow$ p.d.f. $g(x) = \begin{cases} 1/L & \text{if } x \in [0,L] \\ 0 & \text{else.} \end{cases}$

$$f_{X+Y}(x) = (f * g)(x) = \int_{t=-\infty}^{t=\infty} f(t)\, g(x-t)\, dt$$

chge limits to $t \in [0,\infty]$

$$= \int_0^\infty \lambda e^{-\lambda t} g(x-t)\, dt$$

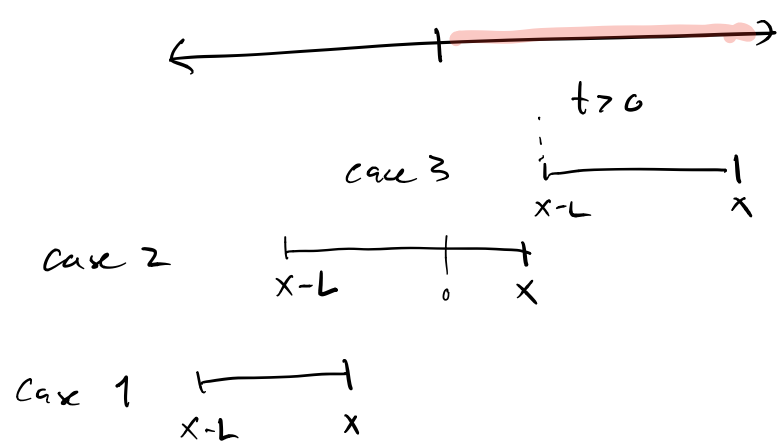
$g(x-t) = 0$ unless

$x - t \in [0,L]$

$0 \le x - t \le L$

$-x \le -t \le L - x$

$$\boxed{x \geq t \geq x - L}$$



1) if $x < 0$ $\int$ ew $t \geq 0$ & $t \in [x-L, x]$
$x < 0$
(not y)

$$f*g(x) = \begin{cases} 0 & \text{if } x<0 \\ & \end{cases}$$

2) if $x - L < 0$, $x > 0$ $(x \in [0, L])$

$$f*g(x) = \int_0^x \lambda e^{-\lambda t} (1/L)\, dt$$

$$= -\frac{e^{-\lambda t}}{L} \Big|_0^x = \frac{1 - e^{-\lambda x}}{L}$$

$$f*g(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{1-e^{-\lambda x}}{L} & \text{if } x \in [0, L] \end{cases}$$

if $x - L > 0$

$$f*g(x) = \int_{x-L}^{x} \lambda e^{-\lambda x} (1/L)\, dx$$

$$= \frac{e^{-\lambda(x-L)} - e^{-\lambda x}}{L}$$

$$f*g(x) = f_{X+Y}(x) = \begin{cases} 0 & \text{if } x<0 \\ \dfrac{1-e^{-\lambda x}}{L} & \text{if } x \in [0,L] \\ \dfrac{e^{-\lambda(x-L)} - e^{-\lambda x}}{L} & \text{if } x > L. \end{cases}$$

$$g*f(x) = \int_{-\infty}^{\infty} g(t) f(x-t)\, dt$$

$$g(t) = \begin{cases} 1/L & \text{if } x \in [0,L] \\ 0 & \text{else} \end{cases}$$

$$= \frac{1}{L}\int_0^L f(x-t)\, dt$$

$$f(x-t) = \begin{cases} 0 & \text{if } x-t \lesssim 0 \\ \lambda e^{-\lambda(x-t)} & \text{if } x-t>0 \\ & (t<x) \end{cases}$$

Recap: Normal distribution &
Γ-distribution

Normal dist. w/ parameters μ, σ^2 is given by p.d.f. $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$

Surprising example

given variables X, Y indep. random, s.t.

joint distribution $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ depends only on $r^2 = x^2 + y^2$

then we find that X & Y are <u>normal</u>.

Consider the variable $X^2 + Y^2$.

Γ-distribution w/ parameters t, λ

p.d.f. $f(x) = c\, e^{-\lambda x} x^{t-1} \longleftrightarrow c e^{-\lambda x} (\lambda x)^{t-1}$

$x \in (0, \infty)$

Calculate constant:

$\int f(x)\,dx = 1$

$$\boxed{f(x) = \frac{\lambda}{\Gamma(t)} e^{-\lambda x} (\lambda x)^{t-1}}$$

$$c \int_0^\infty e^{-\lambda x} (\lambda x)^{t-1}\, dx = \frac{c}{\lambda} \left(\int_0^\infty e^{-u} u^{t-1}\, du \right)$$

$u = \lambda x$

$du = \lambda\, dx$

$$\boxed{\underline{\text{Def}}: \Gamma(t) = \int_0^\infty e^{-u} u^{t-1}\, du}$$

$$\Gamma(1) = \int_0^\infty e^{-u} u^{1-1}\, du = \int_0^\infty e^{-u}\, du = 1$$

$$\Gamma(t) = \int_0^\infty \underset{\substack{\uparrow \\ dw}}{e^{-u}} \underset{\substack{\uparrow \\ v}}{u^{t-1}} du = -e^{-u} \cancel{u^{t-1}} \Big|_0^\infty + \int_0^\infty (t-1) e^{-u} u^{t-2} du$$

0

$$t > 1 \qquad = (t-1)\Gamma(t-1)$$

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)\ldots$$

$$= (n-1)!\,\Gamma(1) = (n-1)!$$

$$\boxed{f(x) = \frac{\lambda}{\Gamma(t)} e^{-\lambda x} (\lambda x)^{t-1}}$$

Γ var. w/ parameters λ, t

$t = 1 \rightarrow f(x) = \lambda e^{-\lambda x}$ <u>exponential variable</u>

Wait time for t occurences in a Poisson process. w/ param λ

Suppose X & Y are independent gamma vars w/ parameters (λ, t), (λ, s)

$$f_{X+Y}(a) = f_X * f_Y(a) = \int_0^a \frac{\lambda}{\Gamma(t)} e^{-\lambda x} (\lambda x)^{t-1} \frac{\lambda}{\Gamma(s)} e^{-\lambda(a-x)} (\lambda(a-x))^{s-1} dx$$

$a - x > 0 \Rightarrow x < a$

$$= C\int_0^a e^{\cancel{-\lambda x}-\lambda a+\cancel{\lambda x}} x^{t-1}(a-x)^{s-1}\,dx$$

$$= C e^{-\lambda a}\int_0^a x^{t-1}(a-x)^{s-1}\,dx$$

$au = x \qquad a\,du = dx$

$$= C e^{-\lambda a}\int_0^1 a^{t-1}u^{t-1}a^{s-1}(1-u)^{s-1}\,a\,du$$

$$= C e^{-\lambda a}a^{t+s-1}\underbrace{\int_0^1 u^{t-1}(1-u)^{s-1}\,du}_{\text{constant}}$$

$$f_{X+Y}(a) = K e^{-\lambda a}a^{t+s-1}$$

$$f_{X+Y}(x) = K e^{-\lambda x}x^{t+s-1}$$

$$= \frac{\lambda}{\Gamma(t+s)}e^{-\lambda x}(\lambda x)^{t+s-1}$$

So $X+Y$ is Γ with params $\lambda, t+s$

X = normal random variable

p.d.f. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$$X^2 \sim \frac{\frac{1}{2} e^{-x/2} (x/2)^{\frac{1}{2}-1}}{\sqrt{\pi}} = \frac{\lambda}{\Gamma(t)} e^{-\lambda x} (\lambda x)^{t-1}$$

$t = \frac{1}{2}$ $\lambda = \frac{1}{2}$

Γ variable w/ $\lambda = \frac{1}{2}$, $t = \frac{1}{2}$

if $X_1, X_2, \ldots, X_n$ indep as above,

$\sum X_i^2$ is Γ w/ params $\lambda = \frac{1}{2}$, $t = \frac{n}{2}$

Chi-squared χ^2 distribution.