Today: 7.1, 7.2

(Properties of expectation)

if X is random variable

discrete: $E[X] = \sum_{x: P(X=x) \neq 0} x P(X=x) = \sum_x x p(x)$

$p_X(x) = p(x) = P(X=x)$

continuous: $E[X] = \int_{-\infty}^{\infty} x f(x) dx$

$f(x) = $ p.d.f.

If $g(x)$ is a (continuous) fcn, $g(X)$ is a random var,

$$E[g(X)] = \sum_{x: P(g(X)=x) \neq 0} x P(g(X)=x) = \sum_{x: P(X=x) \neq 0} g(x) p(x)$$

$X = \{1, -, 6\}$ die roll

$$E[X^2] = 1^2 P(X=1) + 2^2 P(X=2) + 3^2 P(X=3) + \dots$$

$$= 1^2 P(X^2 = 1^2) + 2^2 P(X^2 = 2^2) + \dots$$

$$E[(X-3)^2] = 0\, P((X-3)^2 = 0) + 1\, P((X-3)^2 = 1) + \dots$$

$$= (1-3)^2 P(X=1) + (2-3)^2 P(X=2) + \dots$$

continuous case:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

$f(x)$ p.d.f. for X.

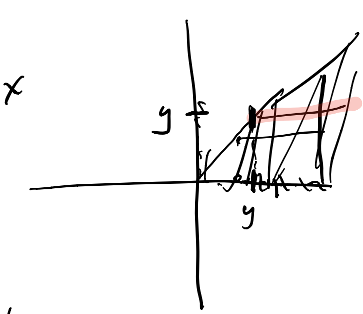
let Y be any random variable w/ p.d.f. $f_Y(x)$

$$E[Y] = \int_{-\infty}^{\infty} x f_Y(x)\,dx$$

$$\int_0^{\infty} x f_Y(x)\,dx = \int_0^{\infty} [x] f_Y(x)\,dx$$

$$= \int_0^{\infty} \left(\int_0^{x} dy \right) f_Y(x)\,dx$$

$$= \int_0^{\infty} \int_0^{x} f_Y(x)\,dy\,dx$$

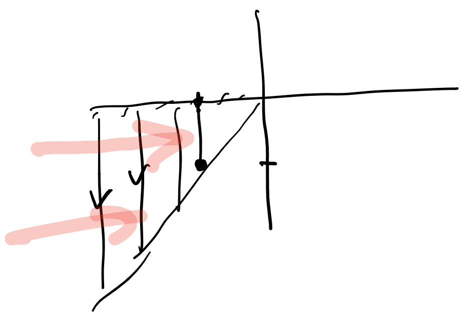


$$= \int_{y=0}^{\infty} \left(\int_{x=y}^{\infty} f_Y(x)\,dx \right) dy$$

$$\underbrace{\qquad\qquad\qquad}_{P(Y>y)}$$

$$\int_0^\infty x f_Y(x)\,dx = \int_{y=0}^{\infty} P(Y>y)\,dy$$

$$\int_{-\infty}^0 x f_Y(x)\,dx = \int_{-\infty}^0 \left(\int_0^x dy\right) f_Y(x)\,dx = -\int_{-\infty}^0 \int_x^0 f_Y(x)\,dy\,dx$$



$$= -\int_{y=-\infty}^0 \underbrace{\int_{-\infty}^y f_Y(x)\,dx}_{P(Y<y)}\,dy$$

$$= -\int_{-\infty}^0 P(Y<y)\,dy$$

$$E[Y] = \int_{-\infty}^\infty x f_Y(x)\,dx = \int_{-\infty}^0 x f_Y(x)\,dx + \int_0^\infty x f_Y(x)\,dx$$

$$\boxed{E[Y] = \int_0^\infty P(Y>y)\,dy - \int_{-\infty}^0 P(Y<y)\,dy}$$

Application:

$$E[g(X)] = \int_0^\infty P(g(X) > y)\,dy - \int_{-\infty}^0 P(g(X) < y)\,dy$$

$$= \int_0^\infty \left(\int_{x: g(x) > y} f(x)\,dx \right) dy - \int_{-\infty}^0 \left(\int_{x: g(x) < y} f(x)\,dx \right) dy$$

$$P(g(X) > y) = \int_{x: g(x) > y} f_X(x)\,dx$$

$$P(X \in A) = \int_{x \in A} f_X(x)\,dx$$

$$P(X < 1) = \int_{-\infty}^1 f_X(x)\,dx$$

$$\iint_{(x,y) \text{ s.t. } g(x) > y,\, y > 0} f(x)\,dx\,dy \quad - \iint_{(x,y) \text{ s.t. } g(x) < y,\, y < 0} f(x)\,dx\,dy$$

$$\int_{x: g(x) > 0} \int_{y=0}^{g(x)} f(x)\,dy\,dx - \int_{x: g(x) < 0} \int_{y=g(x)}^{0} f(x)\,dx\,dy$$

$$= \int_{x: g(x) > 0} f(x) \underbrace{\int_{y=0}^{g(x)} dy}_{g(x)}\,dx + \int_{x: g(x) < 0} f(x) \underbrace{\int_0^{g(x)} dy}_{g(x)}\,dx$$

$$= \int_{x : g(x) > 0} f(x) g(x) dx + \int_{x : g(x) < 0} f(x) g(x) dx$$

$$= \int_{-\infty}^{\infty} f(x) g(x) dx \ !$$:)

<u>Similarly</u> if X, Y are jointly cont. random variables w/ joint dist. fcn $f(x,y)$ then for $g(x,y)$ a function

$$E[g(X,Y)] = \int\int g(x,y) f(x,y) \, dx \, dy$$

"Linearity of expectation"

$$E[X+Y] = \int_y \int_x (x+y) f(x,y) dx dy$$

$$= \int_y \int_x x f(x,y) + y f(x,y) \, dx \, dy$$

$$= \int_y \int_x y f(x,y) dx dy + \int_x \int_y x f(x,y) dy dx$$

$$= \int_y y \int_x f(x,y) dx dy + \int_x x \int_y f(x,y) dy dx$$

$$= \int_y y f_Y(y)\,dy + \int_x x f_X(x)\,dx$$

$$= E[Y] + E[X]$$

"Indicator functions"

given an event $A \subset S$ can define a random variable

$$X = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ doesn't occur} \end{cases}$$

$E[X] = 0 \cdot P(X=0) + 1\,P(X=1)$

$= P(A)$

$$X(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

"indicator vars".

given events $A_1, \dots, A_n$, indicator variables $X_1, \dots, X_n$

then $X = \sum_i X_i$ = "# of events which occur"

n people put hats on a table & pick up random ones

what is the Expected # of people who get their hats back?

A_i - ith person gets their hat.

$X = \sum X_i$ = # people who get their hats.

$$E[X] = \sum_{i=1}^{n} E[X_i] = n \cdot \frac{1}{n} = 1$$

$$E[X_i] = P(A_i) = \frac{1}{n}$$