Given 6 cards labelled 1 through 6, arrange them in a random sequence, let $X = \#$ cards in the correct place

1. $E[X]$
2. $P(X \geq 1)$
3. $P(X = 0)$

1. Let E_i event that the ith card is in the correct place. Let X_i be the corresponding indicator variable.

last time

$X = \sum X_i \qquad E[X_i] = 0 \cdot P(X_i = 0) + 1 P(X_i = 1)$

$= P(X_i = 1) = P(E_i)$

Linearity of expectation: $E[X] = E(\sum X_i) = \sum E[X_i]$

$= \sum_{i=1}^{6} P(E_i) = 6 \cdot \frac{1}{6} = 1$

2. $P(X \geq 1) = P\left(\bigcup_{i=1}^{6} E_i\right)$

$= P(E_1) + P(E_2) + \cdots + P(E_6) - P(E_1 E_2) - \cdots$

$+ P(E_1 E_2 E_3) + \cdots$

$= 6 \cdot P(E_1) - \binom{6}{2} P(E_1 E_2) + \binom{6}{3} P(E_1 E_2 E_3) - \cdots$

$= 6 \frac{1}{6} - \binom{6}{2}\left(\frac{1}{6}\right)\left(\frac{1}{5}\right) + \binom{6}{3}\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)\left(\frac{1}{4}\right) - \cdots$

$$= 1 - \frac{6!}{4!\,2!} \cdot \frac{1}{6}\frac{1}{5} + \frac{6!}{3!\,3!}\,\frac{1}{6}\,\frac{1}{5}\,\frac{1}{4} - \cdots$$

$$\cancel{5!} \qquad \cancel{5!}$$

$$= \boxed{1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}}$$

$$\approx \sum_{i=0}^{\infty} (-1)^i \frac{1}{i!} = e^{-1}$$

3. $P(X=0) = 1 - P(X \geq 1) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$

Example

n men & n women arranged in a circle.
randomly

X = # men standing next to a woman.

$E[X]$ $\qquad$ E_i = event that ith man is next to a woman

X_i - indicator

$X = \sum X_i$

$$E[X] = E[\sum X_i] = \sum E[X_i] = \sum_{i=1}^{n} P(E_i)$$

ith man has two neighbors, chosen randomly from remaing 2n-1 people, n of whom are women, n-1 men. $P(E_i) = 1 - P(\text{both neighbors are men})$

$$P(\text{both men}) = \frac{\binom{n-1}{2}}{\binom{2n-1}{2}}$$

$$E[X] = \sum P(E_i) = \sum_{i=1}^{n} 1 - \frac{\binom{n-1}{2}}{\binom{2n-1}{2}} =$$

$$n\left(1 - \frac{\binom{n-1}{2}}{\binom{2n-1}{2}}\right)$$

if arranged in a line, compute $P(E_i)$ differently depending on whether ith person is on end or not

F_i = event that ith man is in middle.

$$P(E_i) = \underbrace{P(E_i \mid F_i)}_{\text{as before}} \underbrace{P(F_i)}_{\left(\frac{2n-2}{2n}\right)} + \underbrace{P(E_i \mid F_i^c)}_{\frac{n}{2n-1}} \underbrace{P(F_i^c)}_{\left(\frac{2}{2n}\right)}$$

Conditional Expectation

Suppose have a barrel of unfair coins

equally likely to have any prob. of heads from 0 to 1

$X = \#$ flips until 2 heads

$E[X] = ?$

Guess (correct): given p, $X_p = \#$ flips for a coin of type p.

$$E[X_p] = \frac{2}{p}$$

$\int_0^1 E[X_p]\,dp$ diverges.

Philosophy HW: what does this mean?

Start w/ X, Y discrete

<u>Def</u> $p_{X|Y}(x|y) = P(X=x \mid Y=y) = \dfrac{p(x,y)}{p_Y(y)}$

$$= \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$E[X \mid Y=y] = \sum_x x\, P(X=x \mid Y=y)$$

$$= \sum x \, p_{X|Y}(x|y)$$

Ex: flip a coin 10 times get a total of 4 heads

X = # of heads in first 5 flips

expected val of X, given total of 4 heads?

Y = # in second 5 flips, $Z = X + Y$ = total #

$$E[X \mid Z=4] = \sum_{x=0}^{5} x \, P(X=x \mid Z=4)$$

$$P(X=x \mid Z=4) = \frac{P(X=x, Z=4)}{P(Z=4)}$$

$$= \frac{P(X=x, Y=4-x)}{P(Z=4)} \leftarrow \binom{10}{4}\left(\tfrac{1}{2}\right)^{10}$$

$$= \frac{P(X=x)P(Y=4-x)}{\binom{10}{4}\left(\frac{1}{2}\right)^{10}}$$

$$= \frac{\binom{5}{x}\left(\frac{1}{2}\right)^{5}\binom{5}{4-x}\left(\frac{1}{2}\right)^{5}}{\binom{10}{4}\left(\frac{1}{2}\right)^{10}}$$

$$E[X \mid Z=4] = \sum_{x=0}^{4} x \, \frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}}$$

Similarly, can do the continuous case

(next time)