From end of last lecture:

Conditional expectation

X random variable, F event

$E[X|F]$ in discrete case:

definition $\overset{\shortparallel}{\sum_x} x P(X=x|F) = \dfrac{\sum_x x P(X=x, F)}{P(F)}$

Recall: if F_1, F_2 are mutually exclusive & $F_1 \cup F_2 = S$

then $P(A) = P(A|F_1)P(F_1) + P(A|F_2)P(F_2)$

Similarly, can extend this to expected values:

$$E[X] = \sum_x x P(X=x) = \sum_x x\left[P(X=x|F_1)P(F_1) + P(X=x|F_2)P(F_2)\right]$$

$$= \sum_x x P(X=x|F_1)P(F_1) + \sum_x x P(X=x|F_2)P(F_2)$$

$$\boxed{E[X] = E[X|F_1]P(F_1) + E[X|F_2]P(F_2)}$$

Interesting particular case: Y another random variable

Can condition on values of Y:

$$\boxed{E[X] = \sum_y E[X|Y=y]P(Y=y)}$$

$E[X|Y]$ is a random variable depending on Y

$$\boxed{E[X] = E[E[X|Y]]}$$

as a fcn of Y.

main fact of conditional expectation.

<u>Example</u>

Random # of shoppers come into a store, each spends a random amt of money.

X_i - money spent by shopper i.

$X_1, X_2, X_3, \ldots$

M = total money N = # of shoppers.

$$E[M] = E[E[M|N]]$$

$$E[M|N=n] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

$$= n\mu$$

$$E[M] = \sum_n E[M|N=n]P(N=n)$$

$$= \sum_n (n\mu) P(N=n) = \mu \sum_n n P(N=n)$$

$$= \mu E[N]$$

Continuous variables & conditional expectation

$$E[X|Y=y] = \sum_x x P(X=x|Y=y) = \sum_x x \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$P(X=x) = p_X(x) \qquad P(Y=y) = p_Y(y)$$

$$\boxed{p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}}$$

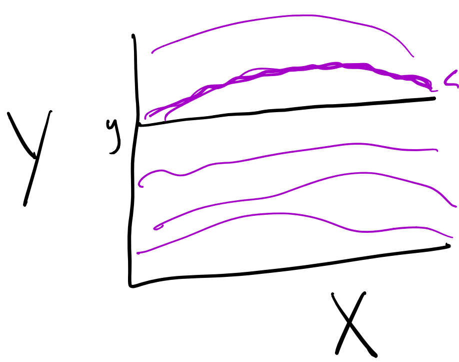
Similarly if X & Y are continuous,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

and $$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

$$= \int_{-\infty}^{\infty} x f(x,y) / f_Y(y) \, dx$$

$$= \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} x f(x,y) \, dx = \frac{\int_{-\infty}^{\infty} x f(x,y) \, dx}{\underbrace{\int_{-\infty}^{\infty} f(x,y) \, dx}_{\text{area under graph at } Y=y}}$$



<u>example</u> $f(x,y) = \begin{cases} \dfrac{e^{-x/y} e^{-y}}{y} & x,y > 0 \\ 0 & \text{else} \end{cases}$

$E[X | Y=y]$

bottom $\int_0^{\infty} \frac{e^{-x/y} e^{-y}}{y} dx = \frac{e^{-y}}{y} \int_0^{\infty} e^{-x/y} dx$

$$= e^{-y}/y \left[-y e^{-x/y} \right]_0^{\infty} = e^{-y}$$

$$\int_0^\infty x e^{-x}\,dx = -xe^{-x}\Big|_0^\infty + \int e^{-x}\,dx$$

$u = x \qquad du = dx$

$dv = e^{-x} \qquad v = -e^{-x}$

$$= -xe^{-x}\Big|_0^\infty - e^{-x}\Big|_0^\infty$$

$$= 1$$

$$\left\{ \begin{matrix} -xe^{-x/y} + y \\ \\ -y \\ \\ y \end{matrix} \right\} \text{?}$$

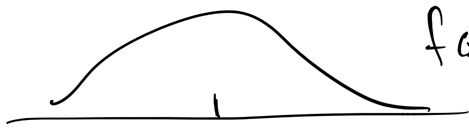
<u>Moments</u>

$X \rightsquigarrow E[X] = \mu_X$

$\rightsquigarrow \mathrm{Var}(X) = E[(X-\mu_X)^2]$

$= E[X^2] - E[X]^2$

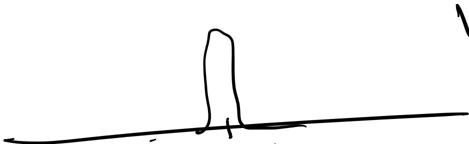
What is the "meaning" of expected vals of powers of a random variable?



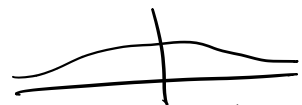
$f(x)$ #1

$\mu_X = E[X]$ "center of mass"

Variance: $Var(X)$ #2

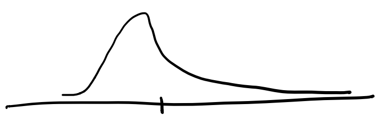


small var



big variance.

"moment of inertia"



#3

skewness

#4

peakedness kurtosis

Problem given events $E_1, E_2, \ldots$

defined indicator variables $I_1, I_2 \ldots$

$$I_i = \begin{cases} 0 & \text{if } E_i^c \\ 1 & \text{if } E_i \end{cases} \qquad X = \sum I_i$$

counts how many happen.

Alternately, could count how many pairs of things happen

$$\sum_{i<j} \underbrace{I_i I_j}_{\text{indicator for } E_i E_j}$$