what's a generating function?

given a sequence of numbers that you care about

$a_0, a_1, a_2, a_3, \ldots$

$\rightsquigarrow f(x) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$

a probability gen fcn:

$a_i = P(X=i)$ for some random var.

$G_X(z)$ = prob. gen fcn for a random var X

$$= \sum_{i=0}^{\infty} P(X=i)\, z^i$$

assume X takes values in $\{0,1,2,\ldots\}$

<u>Properties:</u> $G_X(1) = \sum P(X=i) = 1$

$$G_X(0) = P(X=0)$$

$$G_X'(z) = \sum_{i=0}^{\infty} P(X=i)\, i z^{i-1}$$

$$G_X'(1) = E[X]$$

Main Important Fact: if X & Y are independent

$$G_{X+Y}(z) = G_X(z)\, G_Y(z)$$

Recall: X, Y indep, $E[XY] = E[X]E[Y]$

$$E[XY] = \sum_{x,y} xy\, p(x,y) = \sum_x \left(\sum_y xy\, p(x,y) \right)$$

$$= \sum_y y \left(\sum x\, p_X(x) p_Y(y) \right)$$

$$= \sum_y \left(y p_Y(y) \underbrace{\left(\sum x p_X(x) \right)} \right)$$

$$= \left(\sum x p_X(x) \right) \left(\sum y p_Y(y) \right)$$

$$G_X(z) = \sum_{i=0}^{\infty} P(X=i)\, z^i$$

$$= E\left[z^X\right]$$

if X & Y are indep, $G_{X+Y}(z) = E\left[z^{X+Y}\right]$

$$= E\left[z^X z^Y\right]$$

$$= E[z^X]\,E[z^Y]$$

$$= G_X(z)\,G_Y(z).$$

Bernoulli variable $X = \begin{cases} 1 & \text{prob } p \\ 0 & \text{w/ prob } 1-p \end{cases}$

$$G_X(z) = P(X=0)\,z^0 + P(X=1)\,z^1$$

$$= (1-p) + pz$$

$X = \{1, -, 6\}$ uniform

$$G_X(z) = \frac{1}{6}z^1 + \frac{1}{6}z^2 + \frac{1}{6}z^3 + \frac{1}{6}z^4 + \frac{1}{6}z^5 + \frac{1}{6}z^6$$

$$= \frac{1}{6}\,z\,\frac{1-z^6}{1-z}$$

$$(1-z)(z + z^2 + - z^6) = z - z^7$$

two rolls: $G_X(z) = \left(\frac{1}{6}\,z\,\frac{1-z^6}{1-z}\right)^2$

Remark

X as above

$$G_X''(1) + G_X'(1) - (G'(1))^2 = V(X)$$

"Generatingfunctionology"