## Variance and Covariance (7.4)

**Recall:**

$$Var(X) = E((X - E[X])^2) \qquad \mu_X = E[X]$$

$$= E((X-\mu_X)^2)$$

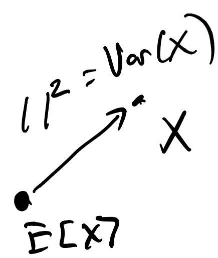
$$E[X^2 - 2XE[X] + E[X]^2]$$

$$= E[X^2] - 2E[X]E[X] + E[E[X]^2]$$

$$= E[X^2] - 2E[X]^2 + E[X]^2$$

$$= E[X^2] - E[X]^2$$

Visual




$$Var(X) = \int (x - \underbrace{E[X]}_{\text{"0"}})^2 f(x)\, dx$$

Recall from "linear algebra"

$$|v|^2 = \langle v, v \rangle$$

**Def** $\mathrm{Cov}(X,Y) = E[(X-E[X])(Y-E[Y])]$

$$= E\{XY - XE[Y] - YE[X] + E[X]E[Y]\}$$

$$E[XE[Y]]$$

$$= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y]$$

$$= E[XY] - E[X]E[Y]$$

**Recall:** If $X$ & $Y$ are independent $\Rightarrow E[XY] = E[X]E[Y]$

$$\Rightarrow \mathrm{Cov}(X,Y) = 0$$

So $\mathrm{Cov}(X,Y) \neq 0 \Rightarrow X$ & $Y$ not independent

**Silly example:** $X$ = uniform on $\{1,-1,0\}$

$$Y = f(X) \quad f(x) = \begin{cases} 0 & \text{if } X \neq 0 \\ 1 & \text{if } X = 0 \end{cases}$$

$$\mathrm{Cov}(X,Y) = \underbrace{E[XY]}_{=0} - \underbrace{E[X]}_{=0}\underbrace{E[Y]}_{=\frac{1}{3}} = 0$$

Facts about Covariance:

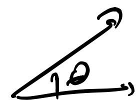
- $\mathrm{Cov}(X,Y) = \mathrm{Cor}(Y,X)$
- $\mathrm{Cov}(X,X) = \mathrm{Var}(X) \geq 0$
- $\mathrm{Cor}(aX+bY, Z) = a\,\mathrm{Cov}(X,Z) + b\,\mathrm{Cov}(Y,Z)$

Partial proof:

$$\mathrm{Cov}(X,X) = E[X \cdot X] - E[X]E[X]$$
$$= E[X^2] - E[X]^2 = \mathrm{Var}(X)$$

$$\mathrm{Cov}(X+Y,Z) = E[(X+Y)Z] - E[X+Y]E[Z]$$
$$= E[XZ+YZ] - \left(E[X]E[Z] + E[Y]E[Z]\right)$$
$$= E[XZ] + E[YZ] - E[X]E[Z] - E[Y]E[Z]$$
$$= \left(E[XZ] - E[X]E[Z]\right) + \left[E[YZ] - E[Y]E[Z]\right]$$
$$= \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$$

$$\langle v, w \rangle = |v||w| \cos\theta$$




$$\cos\theta = \frac{\langle v, w \rangle}{|v||w|}$$

Correlation of $X, Y \equiv \cos\theta$ between them using $\mathrm{Cov}(\ )$ as $\langle , \rangle$

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

"correlation"

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$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Cov}(X+Y, X+Y) \\ &= \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,Y) \\ &= \mathrm{Var}(X) + 2\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)\end{aligned}$$

How to calculate $E[X^2]$ and similar things $E[X^3]$...
for $X$'s we care about.

(7.3)

Suppose given events $E_1, E_2, \ldots, E_n$

$I_i$ indicator for $E_i$, $X = \sum_{i=1}^{n} I_i$

$E[X]$ = # of events we expect to occur.

What if we wanted to know how many pairs of events we expect to occur?

Remark, this is combinatorial reasonable to think about.

$E_iE_j$ has indicator variable $I_iI_j$

So $X^2 = \left(\sum I_i\right)^2 = \sum_{i,j} I_i I_j$

$$= 2\sum_{i<j} I_iI_j + \underbrace{\sum I_i^2}_{X}$$

$$I_i^2 = I_i$$

$$\frac{X^2 - X}{2} = \sum_{i<j} I_i I_j = \sum \text{indicators for } E_i E_j$$

all pairs of events

$$\frac{X(X-1)}{2} = \text{random var for \# of pairs of events which occur.}$$

$$= \binom{X}{2}$$

$$\binom{X}{k} = \text{random var for \# of sets of } k \text{ events which occur.}$$

ex: $X$ is a binomial variable w/ parameters $n, p$

$E[X] = np$

$E[X^2]$

$$E\left[\binom{X}{2}\right] = E\left[\sum_{i,j \text{ pairs}} I_i I_j\right]$$

$E[I_i I_j] = E(I_i) E(I_j)$

$= p^2$

$$= p^2 \binom{n}{2}$$

$$E\left[\frac{X(X-1)}{2}\right] = p^2 \binom{n}{2}$$

$$\frac{1}{2}E[X^2-X] = \frac{1}{2}E[X^2] - \frac{1}{2}\underbrace{E[X]}_{np}$$

$$\parallel$$

$$p^2\binom{n}{2}$$

$$p^2\binom{n}{2} + \frac{np}{2} = \frac{1}{2}E[X^2]$$

$$2p^2\left(\frac{n(n-1)}{2}\right) + np = E[X^2]$$

$$p^2n^2 - p^2n + np = E[X^2]$$

$$Var(X) = E[X^2] - E[X]^2 = np - p^2n$$

$$= np(1-p)$$

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(7.7) Moment generating functions

<u>Def</u> $M_X(t) = E\left[e^{tX}\right]$

i.e. $= \sum_x e^{tx} p(x)$ discrete

$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$ continuous.

Facts: $M'(0) = E[X]$ $\quad M''(0) = E[X^2] \ldots$